Math 1553 Exam 3, Fall 2024

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 7:45 PM on Wednesday, November 13.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If A is a 3×3 matrix and $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then A cannot be invertible.

   ◯ True

   ◯ False

   (b) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

   ◯ True

   ◯ False

   (c) If A is an $n \times n$ matrix and $\det(A) = 3$, then $\lambda = 3$ must be an eigenvalue of A.

   ◯ True

   ◯ False

   (d) Suppose A is a 3×3 matrix with characteristic polynomial

   $$\det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4.$$

   Then the equation $Ax = 0$ must have only the trivial solution.

   ◯ True

   ◯ False

   (e) Suppose A is an $n \times n$ matrix and $\lambda = 2$ is an eigenvalue of A. If u and v are vectors in the 2-eigenspace of A, then $5u - 2v$ must be in the 2-eigenspace of A.

   ◯ True

   ◯ False

2. On this page, you do not need to show work or justify your answers. Parts (a) through (d) are unrelated.

(a) (4 points) Suppose that A is an invertible $n \times n$ matrix. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

◯ Every vector in $\mathbf{R}^n$ is in the span of the columns of A.

◯ A does not have 0 as an eigenvalue.

◯ The RREF of A is the $n \times n$ identity matrix.

◯ The set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is a basis for $\mathbf{R}^n$.

(b) (2 points) Find the area of the parallelogram with vertices

$$(1,1), \quad (2,3), \quad (4,2), \quad (5,4).$$

Clearly fill in the bubble for the correct answer below.

◯ $\frac{1}{2}$ ◯ 1 ◯ 2 ◯ $\frac{5}{2}$ ◯ 5 ◯ 7 ◯ none of these

(c) (2 points) Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Find $\det \begin{pmatrix} a & b & c \\ 3d-g & 3e-h & 3f-i \\ 4d & 4e & 4f \end{pmatrix}$.

Clearly fill in the bubble for the correct answer below.

◯ 2 ◯ -2 ◯ 4 ◯ -4 ◯ 6 ◯ -6

◯ 8 ◯ -8 ◯ 24 ◯ -24 ◯ none of these

(d) (2 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Which **one** of the following is an eigenvector of A? Clearly fill in the bubble for your answer.

◯ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ◯ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ◯ $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ◯ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

3. Multiple choice and short answer. Parts (a)-(d) are unrelated and you do not need to show your work.

(a) (2 points) Find A^{-1} for the matrix $A = \begin{pmatrix} 5 & 3 \\ -2 & 1 \end{pmatrix}$.

Clearly fill in the bubble for your answer below.

◯ $\frac{1}{11}\begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$ ◯ $\begin{pmatrix} -1 & 3 \\ -2 & -5 \end{pmatrix}$ ◯ $\frac{1}{11}\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$ ◯ $\begin{pmatrix} -1 & -3 \\ 2 & -5 \end{pmatrix}$

(b) (2 points) Let $A = \begin{pmatrix} -3 & 1 & -1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Find $\det(A^3)$. Clearly fill in the bubble for your answer below.

◯ -2 ◯ 2 ◯ -6 ◯ 6 ◯ -8 ◯ 8

◯ -1 ◯ 1 ◯ -16 ◯ 16 ◯ none of these

(c) (3 pts) Suppose A is a 2×2 matrix and the vectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ are eigenvectors of A. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

◯ A is invertible.

◯ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ correspond to different eigenvalues of A.

◯ A is diagonalizable.

(d) (3 pts) Let $A = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 7 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 & 0 \\ -3 & 7 & 0 \\ 0 & 2 & 1 \end{pmatrix}^{-1}$.

Which of the following statements must be true? Fill in the bubble for all that apply.

◯ $A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

◯ For each eigenvalue λ of A, the geometric multiplicity of λ is equal to the algebraic multiplicity of λ.

◯ $\det(A - I) = 0$.

4. On this page, you do not need to show your work. Parts (a)-(d) are unrelated.

(a) (2 points) Find the value of a so that $a \cdot \det \begin{pmatrix} 20 & 40 & 16 \\ 2 & 4 & 1 \\ -8 & -2 & 12 \end{pmatrix} = \det \begin{pmatrix} 10 & 20 & 8 \\ -2 & -4 & -1 \\ -4 & -1 & 6 \end{pmatrix}$.

Clearly fill in the bubble for the correct answer below.

◯ $a = 2$ ◯ $a = -2$ ◯ $a = 4$ ◯ $a = 8$ ◯ $a = -4$

◯ $a = -\frac{1}{8}$ ◯ $a = \frac{1}{4}$ ◯ $a = -\frac{1}{4}$ ◯ none of these

(b) (4 points) Suppose A is a 3×3 matrix. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

◯ If v_1, v_2, and v_3 are nonzero vectors in $\mathbf{R}^3$ satisfying $Av_1 = v_1$, $Av_2 = 3v_2$, and $Av_3 = -v_3$, then the set $\{v_1, v_2, v_3\}$ must be linearly independent.

◯ The eigenvalues of A are its diagonal entries.

◯ If $\det(A) = 3$, then $\det(2A^{-1}) = 2/3$.

◯ If A is invertible, then A^7 is invertible.

(c) (2 points) Suppose A is a 2×2 matrix satisfying

$$\text{Nul}(A - 3I) = \text{Span}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\} \quad \text{and} \quad \text{Nul}(A + I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Find $A\begin{pmatrix} 0 \\ -4 \end{pmatrix}$. Hint: $\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Clearly fill in the bubble for the correct answer below.

◯ $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ◯ $\begin{pmatrix} -2 \\ 10 \end{pmatrix}$ ◯ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ◯ $\begin{pmatrix} 2 \\ -10 \end{pmatrix}$ ◯ $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

◯ $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ ◯ $\begin{pmatrix} 4 \\ -8 \end{pmatrix}$ ◯ $\begin{pmatrix} -4 \\ 10 \end{pmatrix}$ ◯ none of these

(d) (2 points) Let A be the 2×2 matrix that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = 3x$. Which **one** of the following vectors is in the (-1)-eigenspace of A? Clearly fill in the bubble for your answer.

◯ $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ◯ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ◯ $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ◯ $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ ◯ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ◯ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5. Short answer and free response. You do not need to show work on (a) and (b). On (c), show your work or you may receive little or no credit, even if your answer is correct.

   (a) (2 points) Is there a diagonalizable 2×2 matrix A whose only eigenvalue is $\lambda = 4$? If your answer is yes, write such an A in the space below. If your answer is no, fill in the bubble labeled "no such A exists."

   $A =$ $\begin{pmatrix} & \\ & \end{pmatrix}$ ◯ no such A exists

   (b) (3 points) Is there a 3×3 triangular matrix B with exactly one eigenvalue $\lambda = 2$, whose 2-eigenspace is a plane? If your answer is yes, write such a B in the space below. If your answer is no, fill in the bubble labeled "no such B exists."

   $B =$ $\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$ ◯ no such B exists

   (c) (5 points) Find A^{-1} for the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Enter your answer below. Show your work on this part.

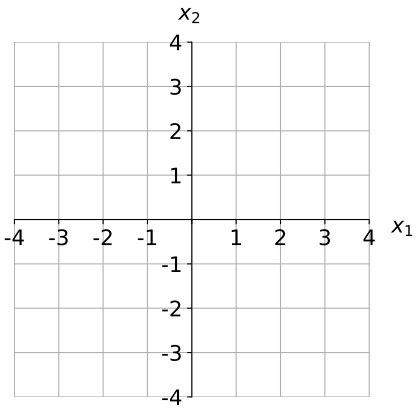
   $A^{-1} =$ $\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

6. Free response. This page is worth 10 points. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $A = \begin{pmatrix} 5 & 2 \\ 4 & -2 \end{pmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace. Draw and label each eigenspace on the graph below.



(c) A is diagonalizable. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$ and write them in the space provided below. You do not need to show your work on this part.

$$C = \begin{pmatrix} & \\ & \end{pmatrix} \qquad D = \begin{pmatrix} & \\ & \end{pmatrix}$$

7. Free response. Show your work! A correct answer without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (4 points) Find $\det(A)$ for $A = \begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 0 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix}$.

Enter your answer here by filling in the blank: $\det(A) =$ _____.

(b) (6 points) Find the complex eigenvalues of the matrix $A = \begin{pmatrix} 1 & -8 \\ 1 & 5 \end{pmatrix}$. For the eigenvalue with **negative** imaginary part, find a corresponding eigenvector v. Simplify your eigenvalues as much as possible!

The eigenvalues are: _____ $v = \begin{pmatrix} \\ \\ \end{pmatrix}$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.