Math 1553 Exam 2, SOLUTIONS, Fall 2024

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The max score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Simplify all fractions and evaluate all trigonometric functions.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam.* If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles to fill in, you must fill them in the correct bubbles clearly and completely or you will not receive credit.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, October 16.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) Suppose v_1, v_2, and v_3 are vectors in $\mathbf{R}^n$ and that there is exactly one solution to the equation
   $$x_1v_1 + x_2v_2 + x_3v_3 = 0.$$
   Then the equation $x_1v_1 + x_2v_2 = 0$ must also have exactly one solution.

   ● True

   ○ False

   (b) Let V be the subspace consisting of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ that satisfy the equation $x - y - 2z = 0$. Then $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for V.

   ● True

   ○ False

   (c) Suppose $T : \mathbf{R}^4 \to \mathbf{R}^3$ and $U : \mathbf{R}^3 \to \mathbf{R}^4$ are linear transformations. Then the composition $U \circ T$ cannot be one-to-one.

   ● True

   ○ False

   (d) If A is a 4×9 matrix, then $\dim(\text{Nul } A) > \dim(\text{Col } A)$.

   ● True

   ○ False

   (e) There is a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ that satisfies the following:
   $$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

   ○ True

   ● False

Solution: Problem 1.

(a) True. There are multiple ways to see this. The fact that $x_1v_1+x_2v_2+x_3v_3=0$ has only one solution means that $\{v_1, v_2, v_3\}$ is linearly independent, so the matrix $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ has 3 pivots (i.e. a pivot in each column). Therefore, the matrix $\begin{pmatrix} v_1 & v_2 \end{pmatrix}$ by itself must have 2 pivots (i.e. a pivot in each column), so $x_1v_1 + x_2v_2 = 0$ has only the trivial solution.

Alternatively, we could visualize it geometrically. If $\{v_1, v_2\}$ is linearly dependent, then $\mathrm{Span}\{v_1, v_2\}$ is at most a line, and therefore $\mathrm{Span}\{v_1, v_2, v_3\}$ would be at most a plane (rather than being 3-dimensional).

(b) True by the Basis Theorem: $V = \mathrm{Nul}\begin{pmatrix} 1 & -1 & -2 \end{pmatrix}$, so we know $\dim(V) = 2$. The two vectors $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ are linearly independent, and they are in the 2-dimensional subspace V since $2 - 0 - 2(1) = 0$ and $3 - 1 - 2(1) = 0$, therefore they are a basis of V by the Basis Theorem.

(c) True: $T(x) = Ax$ for some 3×4 matrix A. Since A is a "wide" matrix, the equation $Ax = 0$ has infinitely many solutions, so $T(x) = 0$ has infinitely many solutions. For any solution to $T(x) = 0$, we have $U(T(x)) = U(0) = 0$ since U is a linear transformation, so the equation $(U \circ T)(x) = 0$ has infinitely many solutions, which means that $U \circ T$ is not one-to-one.

(d) True: Since A is 4×9 it has at most 4 pivots, therefore the equation $Ax = 0$ has at least 5 free variables. This means that the null space is at least 5-dimensional, whereas the column space is at most 4-dimensional.

(e) False. We are told $T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, but if T were linear then we would have

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

2. **Full solutions are on the next page.**

(a) (3 points) Suppose $\{v_1, v_2, v_3\}$ is a basis for a subspace W of $\mathbf{R}^n$. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

● If b is a vector in W, then the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ must have exactly one solution.

● If w_1, w_2, and w_3 are vectors in W and $\text{Span}\{w_1, w_2, w_3\} = W$, then $\{w_1, w_2, w_3\}$ must be a basis for W.

● The set $\{v_1, v_2, v_1 - 2v_2\}$ must be linearly dependent.

(b) (2 points) Suppose A is a 25×20 matrix whose RREF has 7 pivots. Which one of the following describes the null space of A?

● $\text{Nul}(A)$ is a 13-dimensional subspace of $\mathbf{R}^{20}$.

○ $\text{Nul}(A)$ is a 13-dimensional subspace of $\mathbf{R}^{25}$.

○ $\text{Nul}(A)$ is an 18-dimensional subspace of $\mathbf{R}^{20}$.

○ $\text{Nul}(A)$ is an 18-dimensional subspace of $\mathbf{R}^{25}$.

(c) (2 points) Consider the following sets in $\mathbf{R}^2$. Clearly mark the **one** set V that satisfies **all** of the following conditions.

(1) V contains the zero vector.

(2) V is *not* closed under addition.

(3) V is *not* closed under scalar multiplication.

Carefully fill in the bubble for the correct answer.

○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy \geq 0 \right\}$

● $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; y - x^2 \geq 0 \right\}$

○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; y - 2x \geq 0 \right\}$

○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x^2 + y^2 \geq 1 \right\}$

(d) (3 points) Which of the following statements are true? Clearly fill in the bubble for all that apply.

● The set of all solutions to a linear system of 4 homogeneous equations in 5 variables is a subspace of $\mathbf{R}^5$.

○ There is a 5×5 matrix A with the property that $\text{Nul}(A) = \text{Col}(A)$.

○ Suppose that A and B are 2×2 matrices and v is in the null space of A. Then v must be in the null space of AB.

Solution:

(a) (i) True. By definition of basis, the vectors v_1, v_2, and v_3 span W and are linearly independent. Therefore, for any b in W, the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ must be consistent and must have exactly one solution.

(ii) True, by the Basis Theorem. Since $\dim(W) = 3$ we know that any 3 vectors that span W form a basis for W.

(iii) True. The third vector in the set is a linear combination of the first two vectors in the set.

(b) Since A is 25×20, its null space is a subspace of $\mathbf{R}^{20}$. By the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 20, \qquad 7 + \dim(\text{Nul } A) = 20.$$

Therefore, $\dim(\text{Nul } A) = 13$.

(c) The set with condition $xy \geq 0$ is closed under scalar multiplication since if $xy \geq 0$ then $(cx)(cy) = c^2xy \geq 0$.

The set with condition $y - 2x \geq 0$ is closed under addition: it is all points on, and above, the line $y = 2x$.

The set with condition $x^2 + y^2 \geq 1$ does not contain the origin.

The set with condition $y - x^2 \geq 0$ contains the zero vector since $0 - 0 \geq 0$. It is not closed under addition: $\begin{pmatrix}1\\1\end{pmatrix}$ is in V since $1 - 1^2 \geq 0$, but $\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$ which is NOT in V since $2 - 2^2 < 0$. This also shows V is not closed under scalar multiplication since $u = \begin{pmatrix}1\\1\end{pmatrix}$ is in V but $2u$ is not in V.

(d) (i) True: this is just the statement that the null space of a 4×5 matrix is a subspace of $\mathbf{R}^5$.
(ii) Not true: if $\text{Nul}(A) = \text{Col}(A)$, then by the Rank Theorem,

$$5 = \dim(\text{Col } A) + \dim(\text{Nul}\, A) = 2\dim(\text{Col } A),$$

thus $\dim(\text{Col } A) = 2.5$, or in other words A would have 2.5 pivots!
(iii) Not true: if $Av = 0$, why should $A(Bv) = 0$? For example, take $A = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ and $B = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$. Then $v = \begin{pmatrix}0\\1\end{pmatrix}$ is in $\text{Nul}(A)$, but

$$AB\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

3. Multiple choice. Parts (a)-(d) are unrelated and you do not need to show your work. There is no partial credit except on (d).

(a) (2 points) Suppose that $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation. Which **one** of the following conditions guarantees that T is onto?

○ For each x in $\mathbf{R}^n$, there is at least one y in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^n$, there is at most one y in $\mathbf{R}^m$ so that $T(x) = y$.

● For each y in $\mathbf{R}^m$, there is at least one x in $\mathbf{R}^n$ so that $T(x) = y$.

○ For each y in $\mathbf{R}^m$, there is at most one x in $\mathbf{R}^n$ so that $T(x) = y$.

○ For some y in $\mathbf{R}^m$, the equation $T(x) = y$ has infinitely many solutions.

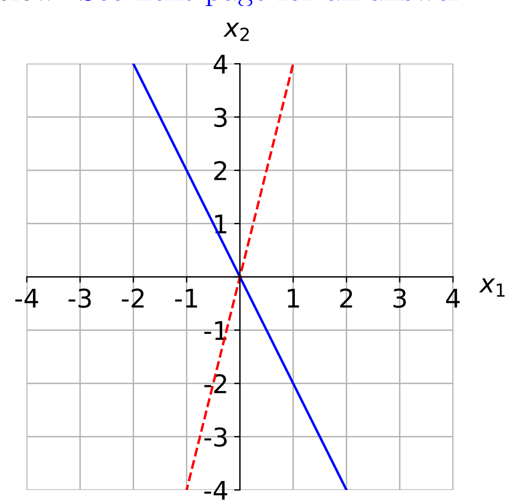
(b) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by $90°$ counterclockwise. Find the vector v so that $T(v) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Carefully fill in the bubble for the correct answer below.

○ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ● $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

(c) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects every vector across the x-axis. What is the matrix for $T \circ U$? Clearly fill in the bubble for the correct answer below.

○ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ● $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(d) (4 points) Write a single matrix A whose column space is the **solid** line below and whose null space is the **dashed** line below. See next page for an answer.



Solution:

(a) T is onto if, for every y in $\mathbf{R}^m$, the equation $T(x) = y$ has at least one solution. This is equivalent to the marked option.

(b) We could do this problem geometrically, by asking ourselves what vector we need to rotate c.c. by 90° to get $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. This vector is $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Alternatively, we could solve $T(v) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ algebraically using the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for counterclockwise rotation by 90°. We solve $Av = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$:

$$\left(\begin{array}{cc|c} 0 & -1 & -1 \\ 1 & 0 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right), \qquad v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c) We need the matrix for $(T \circ U)(x) = T(U(x))$, so we take AB, where A is the matrix for T and B is the matrix for U. The reason for the order is that we first reflect a vector v across the x-axis (this is Bv), then we reflect the result across the line $y = x$, giving us $A(Bv) = ABv$.
Here $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Alternatively, we could just apply $T \circ U$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{acr. x-axis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{acr. y=x}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{acr. x-axis}} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \xrightarrow{\text{acr. y=x}} \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

so the matrix is $\left(T(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(d) $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ and $\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$. Therefore, each column must be a scalar multiple of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and the solution set to $Ax = 0$ satisfies

$$x_1 = \frac{x_2}{4}, \text{ which means } x_1 - \frac{1}{4}x_2 = 0.$$

This means the second column of A must be $-1/4$ times the first column. Many possibilities for A:

$$A = \begin{pmatrix} 1 & -1/4 \\ -2 & 1/2 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1/4 \\ 2 & -1/2 \end{pmatrix}, \qquad A = \begin{pmatrix} 4 & -1 \\ -8 & 2 \end{pmatrix}, \ \textit{etc.}$$

4. (a) (2 points) Suppose A is an $m \times n$ matrix with $m < n$, and let T be the matrix transformation $T(x) = Ax$. Which **one** of the following statements is correct?

○ T must be one-to-one, but T cannot be onto.

○ T cannot be one-to-one, but T must be onto.

● T cannot be one-to-one, but we need more information to determine whether T is onto.

○ T cannot be onto, but we need more information to determine whether T is one-to-one.

(b) (3 points) Which of the following transformations are linear? Clearly fill in the bubble for all that apply.

○ $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x - y - z, x + 1)$.

● $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (3x - 4y, 2x + 5y)$.

● $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (\pi x - \ln(7)y, y - x)$.

(c) (2 points) Consider the matrix below and its reduced row echelon form.

$$A = \begin{pmatrix} 1 & 0 & 3 & 4 \\ -1 & 4 & -11 & -12 \\ -2 & 3 & -12 & -14 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Write a basis for the column space of A in the box provided below.

Many answers possible, for example $\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \right\}$. Any two columns of A will form a basis for Col A in this case, and so will two nonzero linear combinations of columns of A that are not collinear.

(d) (3 points) Suppose A is a matrix with 3 columns v_1, v_2, and v_3 (in that order), and suppose that the vector $r = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ is in the null space of A.

In the **left** box below, write another nonzero vector x (with $x \neq r$) in Nul A.
In the **right** box below, write a linear dependence relation for v_1, v_2, and v_3.
Here, x can be any nonzero scalar multiple of r except for r itself. For example, $x = \begin{pmatrix} -4 \\ 2 \\ 6 \end{pmatrix}$ or $x = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$, etc.

We've been told $Ar = 0$, so $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 0$, which means $\boxed{-2v_1 + v_2 + 3v_3 = 0}$.

Other answers possible, for example any nonzero multiple like $\boxed{-4v_1 + 2v_2 + 6v_3 = 0}$.

Solution:

(a) A is a "wide" matrix, so it cannot have a pivot in every column, therefore $Ax = 0$ must have infinitely many solutions. Therefore, T cannot be one-to-one. However, we need more information to determine whether T is onto. Below we will write an A that gives an onto transformation T, contrasted with an A whose transformation T is not onto.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) (i) Not linear: $T(0,0,0) = (0,1)$ so $T(0) \neq 0$.

(ii) Linear: in fact, $T(x) = Ax$ for $A = \begin{pmatrix} 3 & -4 \\ 2 & 5 \end{pmatrix}$.

(iii) Linear: in fact, $T(x) = Ax$ for $A = \begin{pmatrix} \pi & -\ln(7) \\ -1 & 1 \end{pmatrix}$. Here, π and $\ln(7)$ are just multiplicative constants. The "ln" does not violate linearity because we aren't taking ln of a variable.

(c) Solution is already given on the answer page.

(d) Solution is already given on the answer page.

5. (a) (2 points)

(i) Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be its corresponding transformation $T(x) = Ax$. Which of the following describes T, geometrically? Fill in the bubble for the correct answer.

- ○ Rotation by 90° clockwise
- ○ Rotation by 90° counterclockwise
- ○ Reflection across the x-axis
- ● Reflection across the line $y = -x$

(ii) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors counterclockwise by the angle θ. Which of the following is the standard matrix A for T? Fill in the bubble for the correct answer.

- ● $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
- ○ $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$
- ○ $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
- ○ $A = \begin{pmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix}$

(b) (4 points) Write the standard matrix A for a linear transformation T satisfying all of the following conditions.

i. The range of T is the line $y = -x$ in $\mathbf{R}^2$.

ii. The set of solutions to the equation $T(x) = 0$ is a plane.

Solution: We need $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$, so each column is a mult. of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and we must have exactly one pivot column. We need the solution set to $Ax = 0$ to be a plane, which requires two free variables, so we need exactly two non-pivot columns. Therefore, A has a total of 3 columns. Many possible answers:

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \textit{etc.}$$

(c) (4 pts) Find the values of x and y so that $A^2 = A$ for the matrix $A = \begin{pmatrix} x & y \\ 6 & -4 \end{pmatrix}$.

Enter your answer below: $x = 5$ $\quad y = -\frac{10}{3}$. We set $A^2 = A$:

$$A^2 = \begin{pmatrix} x & y \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x & y \\ 6 & -4 \end{pmatrix} = \begin{pmatrix} x^2 + 6y & xy - 4y \\ 6x - 24 & 6y + 16 \end{pmatrix}, \qquad A = \begin{pmatrix} x & y \\ 6 & -4 \end{pmatrix}.$$

The bottom row gives

- $6x - 24 = 6$, so $6x = 30$ and therefore $\boxed{x = 5}$.
- $6y + 16 = -4$, so $6y = -20$ and thus $\boxed{y = -\frac{10}{3}}$.

With these values, the equations in the first row are also satisfied:
$5^2 + 6(-10/3) = 5$ and $5(-10/3) - 4(-10/3) = -10/3$.

The rest of the exam is free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even if it is correct.

6. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation $T(x, y) = (2x - y, x + 3y, x)$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates vectors $90°$ clockwise.

   (a) Find the standard matrix A for T and write it in the space below. Show your work.

$$A = \left(T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}2 & -1\\ 1 & 3\\ 1 & 0\end{pmatrix}.$$

   (b) Write the standard matrix B for U. Evaluate any trigonometric functions you write. Do not leave your answer in terms of sine and cosine.

$$B = \begin{pmatrix}\cos(90°) & \sin(90°)\\ -\sin(90°) & \cos(90°)\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

   (c) Which composition makes sense: $T \circ U$ or $U \circ T$? Fill in the correct bubble below. You do not need to show your work on this part.

   ● $T \circ U$ ○ $U \circ T$

   Solution: We can do this part without doing (a) or (b). Here $T \circ U$ makes sense: U sends vectors from $\mathbf{R}^2$ to $\mathbf{R}^2$, then T takes the vector from $\mathbf{R}^2$ to $\mathbf{R}^3$.

   On the other hand, $U \circ T$ makes no sense, because if x is a valid input for T, then $T(x)$ lives in $\mathbf{R}^3$ which is not the domain of U, so $U(T(x))$ makes no sense.

   (d) Compute the standard matrix C for the composition you selected in (c). Put your answer in the space provided below.

   Solution: The matrix for $T \circ U$ is AB, so

$$AB = \begin{pmatrix}2 & -1\\ 1 & 3\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & 2\\ -3 & 1\\ 0 & 1\end{pmatrix}.$$

Free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even if it is correct. Parts (a) and (b) are unrelated.

7. (a) (5 points) Let $A = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 5 & 2 \end{pmatrix}$. Find a basis for the null space of A.

Solution: We row-reduce $\left(A \mid 0 \right)$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 1 & 5 & 2 & 0 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right).$$

This gives us $x_1 - 2x_4 = 0$, $x_1 + 7x_4 = 0$, $x_3 - x_4 = 0$, and x_4 free.

$$x_1 = 2x_4, \qquad x_2 = -7x_4, \qquad x_3 = x_4, \qquad x_4 = x_4 \text{ (real)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_4 \\ -7x_4 \\ x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 2 \\ -7 \\ 1 \\ 1 \end{pmatrix}, \text{ so a basis for Nul A is } \left\{ \begin{pmatrix} 2 \\ -7 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(b) (5 points) Find all real values of c (if there are any) so that the set below is linearly independent:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} c \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Solution: We row-reduce the matrix A whose three columns are these vectors, and determine when A has a pivot in every column.

$$\begin{pmatrix} 1 & 2 & c \\ -1 & 1 & 1 \\ 3 & 7 & 2 \end{pmatrix} \xrightarrow[R_3 = R_1 - 3R_3]{R_2 = R_2 + R_1} \begin{pmatrix} 1 & 2 & c \\ 0 & 3 & 1+c \\ 0 & 1 & 2-3c \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & c \\ 0 & 1 & 2-3c \\ 0 & 3 & 1+c \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 3R_2} \begin{pmatrix} \boxed{1} & 2 & c \\ 0 & \boxed{1} & 2-3c \\ 0 & 0 & \boxed{1+c-6+9c} \end{pmatrix}.$$

Since the bottom right entry must be pivot, it must be nonzero:

$$1 + c - 6 + 9c \neq 0, \qquad 10c \neq 5, \qquad c \neq \frac{1}{2}.$$

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