# Math 1553 Exam 2, Spring 2025

| **Name** | | **GT ID** | |
|---|---|---|---|

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A+HP, 8:25-9:15 AM)        Jankowski (C, 9:30-10:20 AM)

Al Ahmadieh (I, 2:00-2:50 PM)        Al Ahmadieh (M, 3:30-4:20 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.

- There are no calculators or aids of any kind (notes, text, etc.) allowed.

- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.

- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!

- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.

- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam.* If you do this, you must clearly indicate it.

- You may cite any theorem proved in class or in the sections we covered in the text.

- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

*I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, March 5.*

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If $\{v_1, v_2, v_3, v_4\}$ is a linearly dependent set of vectors in $\mathbf{R}^n$, then $v_4$ must be in $\mathrm{Span}\{v_1, v_2, v_3\}$.

      ○ True

      ○ False

   (b) If $A$ is a $4 \times 3$ matrix and the columns of $A$ are linearly independent, then $\dim(\mathrm{Col}\ A) = 3$.

      ○ True

      ○ False

   (c) Suppose $A$ is a $5 \times 4$ matrix and $b$ is a vector so that $Ax = b$ is consistent. Then the set of solutions to $Ax = b$ must be a subspace of $\mathbf{R}^4$.

      ○ True

      ○ False

   (d) Suppose $A$ is an $m \times n$ matrix with corresponding matrix transformation $T(x) = Ax$. If the RREF of $A$ has a row of zeros, then $T$ cannot be one-to-one.

      ○ True

      ○ False

   (e) Suppose $A$ is an $n \times n$ matrix and there is a vector $b$ in $\mathbf{R}^n$ so that the equation $Ax = b$ has exactly one solution. Then $A$ must be invertible.

      ○ True

      ○ False

2. On this page, you do not need to show work and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (3 points) Suppose $v_1$, $v_2$, and $v_3$ are vectors in $\mathbf{R}^n$. Which of the following statements are true? Fill in the bubble for all that apply.

○ If $\{v_1, v_2, v_3\}$ is linearly independent, then the only solution to the vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ is: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

○ If $\{v_1, v_2, v_3\}$ is linearly independent and $b$ is a vector with the property that the vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = b$ is consistent, then the vector equation must have exactly one solution.

○ If $\{v_1, v_2, v_3\}$ is linearly dependent, then $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ for some scalars satisfying $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$.

(b) (2 pts) Find all values of $h$ so that the following vectors form a basis for $\mathbf{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -2 \\ h \end{pmatrix}.$$

○ $h = 0$ only     ○ $h = -1$ only     ○ $h = 1$ only     ○ $h = -2$ only

○ all $h$ except $0$     ○ all $h$ except $-1$     ○ none of these

(c) (2 points) Suppose $A$ is a $40 \times 20$ matrix and the solution set to $Ax = 0$ is 5-dimensional. Which **one** of the following describes the column space of $A$? Fill in the bubble for your answer.

○ $\mathrm{Col}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{20}$.

○ $\mathrm{Col}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{40}$.

○ $\mathrm{Col}(A)$ is a 5-dimensional subspace of $\mathbf{R}^{20}$.

○ $\mathrm{Col}(A)$ is a 5-dimensional subspace of $\mathbf{R}^{40}$.

(d) (3 points) Suppose $A$ is a $2 \times 2$ matrix, and let $V$ be the set of all vectors $x$ in $\mathbf{R}^2$ that satisfy $Ax = 3x$. Which of the following statements are true? Fill in the bubble for all that apply.

○ $V$ contains the zero vector.

○ $V$ closed under addition? In other words, if $u$ and $v$ are vectors in $V$, must it be true that $u + v$ is in $V$?

○ $V$ closed under scalar multiplication? In other words, if $u$ is a vector in $V$ and $c$ is a scalar, must it be true that $cu$ is in $V$?

3. On this page, you do not need to show work and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Which **one** of the following linear transformations is onto? Clearly fill in the bubble for your answer.

○ $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x, y) = (x, x - y, 2x + y)$.

○ $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x - y - z,\ 3x - 3y - 3z)$.

○ $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (x - y,\ 2x - y)$.

○ $T : \mathbf{R}^3 \to \mathbf{R}^3$ that projects vectors onto the $xy$-plane.

(b) (2 points) Let $A$ be an $m \times n$ matrix, and let $T$ be the corresponding matrix transformation $T(x) = Ax$. Which **one** of the following statements guarantees that $T$ is one-to-one?

○ For each $x$ in $\mathbf{R}^n$, there is at least one $y$ in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each $x$ in $\mathbf{R}^n$, there is at most one $y$ in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each $y$ in $\mathbf{R}^m$, there is at least one $x$ in $\mathbf{R}^n$ so that $T(x) = y$.

○ For each $y$ in $\mathbf{R}^m$, there is at most one $x$ in $\mathbf{R}^n$ so that $T(x) = y$.

(c) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by $90°$ **clockwise**. Solve $T(x) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Fill in the bubble for your answer below.

○ $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$    ○ $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$    ○ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$    ○ $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$    ○ none of these

(d) (4 points) Suppose $A$ is a $7 \times 9$ matrix and $B$ is a $9 \times 4$ matrix, and let $T$ be the matrix transformation $T(x) = ABx$.

(i) What is the domain of $T$?    ○ $\mathbf{R}^4$    ○ $\mathbf{R}^7$    ○ $\mathbf{R}^9$

(ii) What is the codomain of $T$?    ○ $\mathbf{R}^4$    ○ $\mathbf{R}^7$    ○ $\mathbf{R}^9$

(iii) Which of the following statements are true? Fill in the bubble for all that apply.

○ If $x$ is in the domain of $T$, then $ABx$ must be in the column space of $B$.

○ Every vector in the null space of $B$ must also be in the null space of $AB$.

4. On this page, you do not need to show your work. Only your answers are graded. Parts (a)-(d) are unrelated.

(a) (2 points) Suppose $A$ is an invertible $n \times n$ matrix. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

○ The RREF of $A$ is the $n \times n$ identity matrix.

○ If $B$ is an invertible $n \times n$ matrix, then $(AB)^{-1} = A^{-1}B^{-1}$.

(b) (2 points) Let $A = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$. Find $A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Fill in the bubble for your answer.

○ $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$   ○ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$   ○ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$   ○ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$   ○ $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$
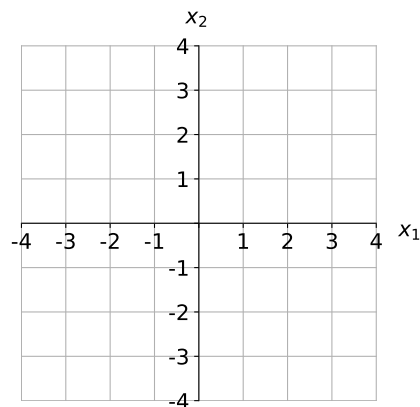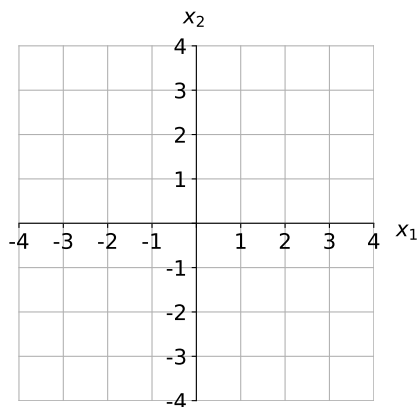
○ $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$   ○ $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$   ○ $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$   ○ $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$   ○ $\begin{pmatrix} -5 \\ -2 \end{pmatrix}$

(c) (2 pts) Write one specific subspace $V$ of $\mathbf{R}^3$ that contains $\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$.

(d) (4 points) Let $A = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$.
On the **left** graph below, draw $\mathrm{Col}(A)$ carefully.
On the **right** graph below, draw $\mathrm{Nul}(A)$ carefully.

5. Free response. Show your work! A correct answer without appropriate work will receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x \\ y - 3x \end{pmatrix}$.

Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by $90°$ **counter**clockwise.

(a) (3 points) Find the standard matrix $A$ for $T$ and write it in the space below. Show your work.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

(b) (2 points) Write the standard matrix $B$ for $U$ in the space below. Evaluate all trigonometric functions you write. Do not leave your answers in terms of sine and cosine.

$$B = \begin{pmatrix} & & \\ & & \end{pmatrix}$$

(c) (1 point) Which composition makes sense: $T \circ U$ or $U \circ T$? Fill in the correct bubble below. You do not need to show your work on this part.
○ $T \circ U$        ○ $U \circ T$

(d) (4 points) Compute the standard matrix $C$ for the composition you circled in part (c). Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

6. Free response. Show your work on (b) and (d), where a correct answer without sufficient work may receive little or no credit. You not need to show your work on (a) or (c). For this page, consider the matrix $A$ and its reduced row echelon form given below.

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (2 points) Write a basis for Col $A$. (no work required on this part)

(b) (4 points) Find a basis for Nul $A$.

(c) (2 points) Define a linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^4$ by $T(x) = Ax$.

- Is $T$ one-to-one?     ○ Yes          ○ No

- Is $T$ onto?          ○ Yes          ○ No

(d) (2 points) Is $\begin{pmatrix} 0 \\ -3 \\ -3 \\ 3 \end{pmatrix}$ in the span of the columns of $A$? Justify your answer.

7. Free response. Show your work except on part (c)! A correct answer without sufficient work in (a) or (b) will receive little or no credit. Parts (a) through (c) are unrelated.

(a) (3 points) Suppose $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ is a linear transformation satisfying:

$$T\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Find $T\begin{pmatrix} -2 \\ 3 \\ -8 \end{pmatrix}$. Hint: $\begin{pmatrix} -2 \\ 3 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.

Enter your answer here: $\begin{pmatrix} \\ \end{pmatrix}$

(b) (4 points) Let $A = \begin{pmatrix} x & y \\ 4 & -2 \end{pmatrix}$. Find all values of $x$ and $y$ so that $A^2 = A$.

Enter your answer here: $x = $ _____ $y = $ _____.

(c) (3 points) Write a single matrix $A$ whose corresponding linear transformation $T$ (given by $T(x) = Ax$) satisfies both of the following conditions.

- The solution set to $T(x) = 0$ is a line.

- The range of $T$ is the plane in $\mathbf{R}^3$ spanned by $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} \\ \\ \end{pmatrix}.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.