Math 1553 Examination 1, Fall 2024

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The max score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam.* If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles to fill in, you must fill in the correct bubbles clearly and completely or you will not receive credit.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, September 18.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If a consistent system of linear equations has more variables than equations, then the system must have infinitely many solutions.

   ◯ True

   ◯ False

   (b) If the RREF of an augmented matrix has bottom row $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 4 \end{array}\right)$, then its corresponding system of linear equations must be consistent.

   ◯ True

   ◯ False

   (c) There is a 2×2 matrix A so that the set of solutions to $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the line $y = x - 1$.

   ◯ True

   ◯ False

   (d) Suppose u, v, and w are vectors in $\mathbf{R}^3$ and $\text{Span}\{u, v, w\} = \mathbf{R}^3$. If b is any vector in $\mathbf{R}^3$, then the vector equation $x_1u + x_2v + x_3w = b$ must have exactly one solution.

   ◯ True

   ◯ False

   (e) If A is an $m \times n$ matrix and $m > n$, then the homogeneous matrix equation $Ax = 0$ must have exactly one solution.

   ◯ True

   ◯ False

2. Multiple choice and short answer. You do not need to show work or justify your answers. Parts (a) through (d) are unrelated. Parts (a) and (d) are worth 3 points each, while parts (b) and (c) are worth 2 points each.

(a) Which of the following matrices are in reduced row echelon form (RREF)? Clearly mark all that apply.

○ $\left(\begin{array}{cccc|c} 1 & 6 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$ ○ $\left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ ○ $\left(\begin{array}{cccc|c} 1 & -2 & 0 & 7 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

(b) Suppose an augmented matrix with 3 rows and 5 columns (including its rightmost column) has 3 pivots and corresponds to a consistent system of linear equations. Clearly mark your answer for each of the following.

(i) The solution set is:
○ a point ○ a line ○ a plane ○ $\mathbf{R}^3$ ○ $\mathbf{R}^4$

(ii) The solution set lives in:
○ $\mathbf{R}$ ○ $\mathbf{R}^2$ ○ $\mathbf{R}^3$ ○ $\mathbf{R}^4$ ○ $\mathbf{R}^5$

(c) In each case, determine whether the equation in the variables x, y, and z is linear or not linear. Clearly mark your answers.

$-2x + 3^{1/2}y + z = 4.$ ○ Linear ○ Not linear

$x - xy - z = 3.$ ○ Linear ○ Not linear

(d) Suppose that the solution set to a matrix equation $Ax = b$ has parametric form

$$x_1 = 3 - x_3, \qquad x_2 = 1 + x_3, \qquad x_3 = x_3 \ (x_3 \text{ real}), \qquad x_4 = 0.$$

Which of the following statements **must** be true? Clearly mark all that apply.

○ The vector $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is a solution to the homogeneous matrix equation $Ax = 0$.

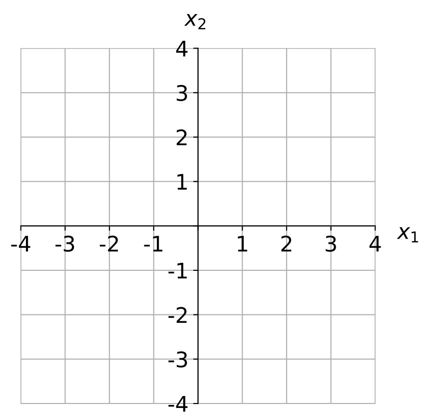
○ The matrix A has 4 columns.

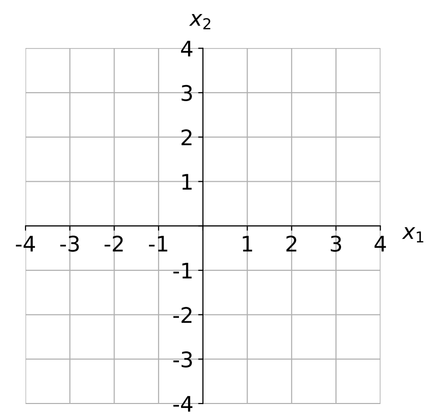
○ b is a vector in $\mathbf{R}^4$.

3. Short answer and multiple choice. You do not need to show your work on this problem. Parts (a) through (c) are unrelated. There is no partial credit except on (a).

(a) (4 points) For some 2×2 matrix A, the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is in the span of the columns of A, and the vector $x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is a solution to $Ax = 0$.

On the **left** graph below, carefully draw the span of the columns of A.
On the **right** graph below, carefully draw the solution set for $Ax = 0$.





(b) (2 points) Compute the product $\begin{pmatrix} 2 & -4 & -3 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

Select the correct answer below.

○ $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$

○ $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 4 & 4 & 0 \\ 4 & -1 & 0 \end{pmatrix}$ ○ none of these

(c) (4 points) Suppose v_1, v_2, v_3, and b are vectors in $\mathbf{R}^n$ and that the augmented matrix $\left(\begin{array}{ccc|c} v_1 & v_2 & v_3 & b \end{array}\right)$ corresponds to a linear system of equations with infinitely many solutions. Which of the following statements are true? Clearly mark all that apply.

○ The vector b is in $\text{Span}\{v_1, v_2, v_3\}$.

○ If A is the matrix with columns v_1, v_2, and v_3, then the span of the columns of A is $\mathbf{R}^3$.

○ The vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ has infinitely many solutions.

○ There are scalars x_1, x_2, and x_3 so that $x_1 v_1 + x_2 v_2 + x_3 v_3 = b$.

4. Short answer. You do not need to show your work on this page. Parts (a) through (d) are unrelated. Parts (a) and (b) are worth 2 points each, but parts (c) and (d) are worth 3 points each.

(a) Consider the system of linear equations corresponding to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right)$. Which one of the following vectors is a solution to the system?

○ $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ ○ $\begin{pmatrix} -13 \\ -3 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix}$ ○ none of these

(b) In each case, determine if the span of the columns of the matrix is $\mathbf{R}^2$. Clearly mark your answer.

(i) $\begin{pmatrix} 4 & -8 & 1 \\ 8 & -16 & 2 \end{pmatrix}$ ○ Yes ○ No

(ii) $\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 5 \end{pmatrix}$ ○ Yes ○ No

(c) Suppose b is a vector in $\mathbf{R}^2$. Clearly mark which **one** of the following statements must be true about the vector equation: $x_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = b.$

○ The vector equation must have exactly one solution, no matter what b is.

○ We cannot conclude whether the vector equation is consistent unless we know the vector b.

○ The vector equation must be consistent, but we cannot conclude whether it has one solution or infinitely many solutions unless we know what b is.

(d) Suppose that A is an $m \times n$ matrix and b is a vector so that the matrix equation $Ax = b$ is consistent. Which of the following statements are true? Clearly mark all that apply.

○ If x is a solution to $Ax = b$, then x is a vector in $\mathbf{R}^m$.

○ The equation $Ax = b$ must be homogeneous if the zero vector is a solution.

○ If the solution set to $Ax = b$ is a line, then the solution set to $Ax = 0$ must also be a line.

The rest of the exam is free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even it is correct. Parts (a) and (b) are unrelated.

5. (a) (5 pts) Solve the following linear system in the variables x_1, x_2, and x_3:

$$x_1 - x_2 - x_3 = 5$$

$$-x_1 + 2x_2 + 4x_3 = 0.$$

Write the solution set in parametric form.

(b) (5 points) Find all values of h and k so that the following system has exactly one solution:

$$2x - hy = 8$$

$$8x + 16y = k.$$

Free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even it is correct.

6. Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$\begin{aligned} x_1 - 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 6x_2 + 3x_3 + 2x_4 &= 3 \\ -3x_1 + 9x_2 - 3x_3 - 2x_4 &= -8. \end{aligned}$$

(a) (5 points) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

(b) (4 points) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

(c) (1 pt) Write *one* vector x that is not the zero vector and solves the corresponding **homogeneous** system below. There is no work required and no partial credit, so take time to check by hand that your answer is correct, and if it is not correct then go back and check your work from above!

$$\begin{aligned} x_1 - 3x_2 + x_3 + x_4 &= 0 \\ 2x_1 - 6x_2 + 3x_3 + 2x_4 &= 0 \\ -3x_1 + 9x_2 - 3x_3 - 2x_4 &= 0. \end{aligned} \qquad x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct. Parts (a) and (b) are unrelated.

7. (a) Find all values of h (if there are any) so that the vector $\begin{pmatrix} -1 \\ h \\ 5 \end{pmatrix}$ can be written as a linear combination of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

(b) Write an augmented matrix in reduced row echelon form, so that the set of solutions to the corresponding system of equations has parametric form given below. You do not need to show your work on this part.

$$x_1 = 5x_3, \qquad x_2 = 0, \qquad x_3 = x_3 \ (x_3 \text{ real}).$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.