MATH 1553, EXAM 3
SPRING 2024

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, April 10.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points. As stated in the instructions, the entries of all matrices on this exam are assumed to be real numbers unless stated otherwise.

a) If A and B are $n \times n$ matrices, then

$$\det(A+B) = \det(A) + \det(B).$$

TRUE FALSE

b) Suppose A is an $n \times n$ matrix and v is an eigenvector of A. Then $-2v$ is also an eigenvector of A.

TRUE FALSE

c) Consider the linear transformation $T(x) = Ax$ where $A = \begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix}$. If S is a circle in $\mathbf{R}^2$ whose area is 4, then the area of $T(S)$ is 32.

TRUE FALSE

d) If A is a diagonalizable $n \times n$ matrix, then every nonzero vector in $\mathbf{R}^n$ is an eigenvector of A.

TRUE FALSE

e) If A is a 3×3 matrix, then A must have exactly two complex eigenvalues.

TRUE FALSE

2. Parts (a) through (d) are unrelated. You do not need to show your work on this page, and there is no partial credit.

a) (4 points) Suppose A is a 3×3 matrix. Which of the following statements must be true? Clearly circle all that apply.

(i) If $\det(A) \neq 0$, then A is invertible.

(ii) If B is a 3×3 matrix, then

$$\det(AB) = \det(A)\det(B).$$

(iii) If A is invertible, then $\det(A^{-1}) = -\det(A)$.

(iv) If v and w are eigenvectors of A satisfying $Av = 8v$ and $Aw = 3w$, then $v + w$ **cannot** be an eigenvector of A.

b) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Find

$$\det\begin{pmatrix} 2d-3a & 2e-3b & 2f-3c \\ a & b & c \\ g & h & i \end{pmatrix}.$$

Clearly circle your answer below.

(i) 1/2 (ii) −1/2 (iii) 2 (iv) −2 (v) 4

(vi) −4 (vii) 6 (viii) −6 (ix) −12 (x) none of these

c) (2 points) Find the value of a so that

$$a \cdot \det\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 5 & 3 & 10 \end{pmatrix} = \det\begin{pmatrix} 2 & -2 & 4 \\ 0 & 2 & 2 \\ 10 & 6 & 20 \end{pmatrix}.$$

(i) $a = 1/8$ (ii) $a = 1/4$ (iii) $a = 1/2$ (iv) $a = 1$ (v) $a = 2$

(vi) $a = 4$ (vii) $a = 8$ (viii) −12 (ix) none of these

d) (2 points) Find the area of the triangle with vertices $(1,1)$, $(2,3)$, and $(3,-2)$. Clearly circle your answer below.

(i) 1 (ii) 1/2 (iii) 3 (iv) 5 (v) 5/2 (vi) 7 (vii) 7/2

3. Short answer and multiple choice. Parts (a) through (c) are unrelated. You do not need to show your work on this page, and there is no partial credit.

a) (3 points) Suppose x is a **nonzero** vector in $\mathbf{R}^n$ and A is an $n \times n$ matrix. Which of the following conditions guarantee that x is an eigenvector of A? Clearly circle all that apply.

(i) x is in the null space of A.

(ii) Ax is a scalar multiple of x.

(iii) The set $\{x, Ax\}$ is linearly independent.

b) (3 points) Let A be an $n \times n$ matrix so that $\det(A - 2I) = 6$. Which of the following statements **must** be true? Clearly circle all that apply.

(i) $\lambda = 2$ is an eigenvalue of the matrix A.

(ii) The only solution to $(A - 2I)x = 0$ is the trivial solution $x = 0$.

(iii) The matrix A is invertible.

c) (4 points) Let C be a 3×3 matrix with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, and suppose that the 2-eigenspace of C is $\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

(i) Find $C\begin{pmatrix}1\\0\\1\end{pmatrix}$. Clearly circle your answer below.

(I) $\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ (II) $\begin{pmatrix}2\\0\\2\end{pmatrix}$ (III) $\begin{pmatrix}4\\0\\4\end{pmatrix}$ (IV) $\begin{pmatrix}2\\0\\0\end{pmatrix}$ (V) Not enough information

(ii) Must it be true that C is diagonalizable?

YES NO NOT ENOUGH INFORMATION

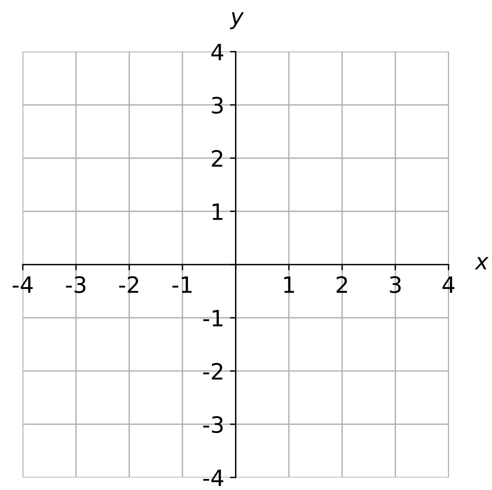
(iii) Which of the following vectors **could** be in the (-1)-eigenspace of C? Clearly circle all that apply.

(I) $\begin{pmatrix}4\\0\\5\end{pmatrix}$ (II) $\begin{pmatrix}6\\1\\1\end{pmatrix}$ (III) $\begin{pmatrix}0\\0\\3\end{pmatrix}$ (IV) $\begin{pmatrix}2\\-7\\5\end{pmatrix}$.

4. Short answer. Parts (a) through (c) are unrelated. There is no work necessary on (a) and (b), but show your work on part (c).

a) (4 points) Let A be the matrix that reflects vectors in $\mathbf{R}^2$ across the line $y = -2x$.
(i) Write the eigenvalues of A in the space below.

(ii) Draw two linearly independent eigenvectors of A on the graph below. For this problem, you only need to draw the eigenvectors (you do not need to label which eigenspace they are in).



b) (2 points) Which **one** of the following matrices A has a 3-eigenspace that is a plane? Clearly circle your answer.

(i) $A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 3 & 6 & 3 \\ 0 & 3 & 0 \\ 0 & 6 & 6 \end{pmatrix}$

c) (4 points) Let $A = \begin{pmatrix} 0.4 & 0.1 \\ 0.6 & 0.9 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 100 \\ 40 \end{pmatrix}$ approach as n gets very large? Write your answer in the space below. Show your work on this part, and fully simplify any fractions that are in your final answer.

$\begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Free response. Show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct.

5. Let $A = \begin{pmatrix} 2 & 0 & -6 \\ 3 & 5 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

a) Find all eigenvalues of A.

b) Find a basis for each eigenspace of A.

c) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

Free response. Show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct. Parts (a) and (b) are unrelated.

6. **a)** Find **all** the values of c such that the matrix

$$A = \begin{pmatrix} 4 & 8 & 0 \\ 2 & c & 0 \\ 1 & 6 & c \end{pmatrix}$$

satisfies $\det(A) = 20$.

b) Find the complex eigenvalues of

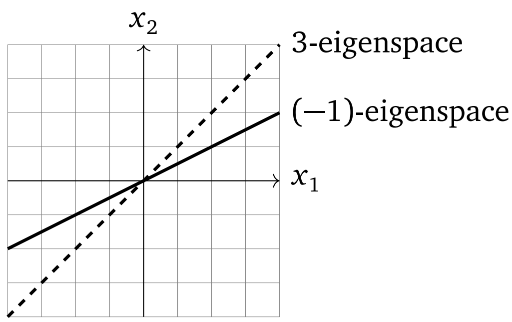
$$A = \begin{pmatrix} 4 & -1 \\ 17 & 2 \end{pmatrix}.$$

Fully simplify your answer for the eigenvalues. For the eigenvalue with **negative** imaginary part, write one eigenvector v. Enter your answers below.

The eigenvalues are: [ ] $v = \begin{pmatrix} \quad \end{pmatrix}$

Free response. Show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct. Parts (a) and (b) are unrelated.

7. **a)** (5 points) Find the matrix A whose (-1)-eigenspace is the **solid** line below and whose 3-eigenspace is the **dashed** line below.
(Note: each square in the grid has sides of length 1)



Enter your answer here: $A = \begin{pmatrix} & \\ & \end{pmatrix}$

b) (5 points) Axel and Billy are magicians who compete for customers in a group of 200 people. Today, Axel has 160 customers and Billy has 40 customers. Each day:

- 30% of Axel's customers keep attending Axel's show, while 70% of Axel's customers switch to Billy's show.
- 80% of Billy's customers attend Billy's show, while 20% of Billy's customers switch to Axel's show.

(i) Write a positive stochastic matrix B and a vector x so that Bx will give the number of customers for Axel's show and Billy's show (in that order) tomorrow. You do not need to compute Bx.

$B = \begin{pmatrix} & \\ & \end{pmatrix}$ $\quad x = \begin{pmatrix} \\ \end{pmatrix}$

(ii) Find the steady-state vector w for B. Write your answer in the space below.

$w = \begin{pmatrix} \\ \end{pmatrix}$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.