MATH 1553, EXAM 3
SPRING 2023

Name		GT ID	

Circle your lecture below.

Jankowski, lec. A and HP (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Sane, lecture G (12:30-1:20 PM)

Sun, lecture I (2:00-2:50 PM) Sun, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, April 12.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose S is a rectangle in $\mathbf{R}^2$ with area 2, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the matrix transformation

$$T(x) = \begin{pmatrix} 1 & 0 \\ 15 & 3 \end{pmatrix} x.$$

Then the area of $T(S)$ is 6.

TRUE FALSE

b) Suppose that $\lambda = -5$, $\lambda = 1$, and $\lambda = 8$ are eigenvalues of a 5×5 matrix A. If $\dim(\mathrm{Null}(A-8I)) = 3$, then A must be diagonalizable.

TRUE FALSE

c) There is an $n \times n$ matrix A so that the zero vector is an eigenvector of A.

TRUE FALSE

d) Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Then $\lambda = 0$ is an eigenvalue of A.

TRUE FALSE

e) Let A be the 2×2 matrix that rotates vectors in $\mathbf{R}^2$ by 65 degrees counterclockwise. Then A has no real eigenvalues. TRUE FALSE

Problem 2.

Parts (a), (b), (c), and (d) are unrelated. On (a) and (b), you do not need to show your work, and there is no partial credit. Show your work on (c) and (d).

a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 7$. Find $\det\begin{pmatrix} d & e & f \\ a & b & c \\ -2a+g & -2b+h & -2c+i \end{pmatrix}$.

Clearly circle the correct answer below.

(i) 7 (ii) -7 (iii) 14 (iv) -14

(v) 56 (vi) -56 (vii) not enough information (viii) none of these

b) (2 points) Write a 2×2 matrix A that is diagonalizable but not invertible.

Enter your answer here: $A = \begin{pmatrix} & \\ & \end{pmatrix}$

c) (3 points) Find the area of the triangle with vertices $(1,2)$, $(2,3)$, and $(4,-5)$.

d) (3 points) Find all values of a so that $\det\begin{pmatrix} 1 & -3 & 0 \\ 1 & a & 0 \\ a & 0 & a \end{pmatrix} = 0$.

Problem 3.

Parts (a), (b), and (c) are unrelated. You do not need to show your work on this page, and there is no partial credit.

a) (3 points) Let A and B be 3×3 matrices satisfying $\det(A) = 2$ and $\det(B) = -3$. Which of the following must be true? Clearly circle all that apply.

(i) $\det(A+B) = \det(A) + \det(B)$.

(ii) $\det(A^T B^{-1}) = -2/3$.

(iii) $\det(-2A) = -16$.

b) (4 points) Suppose A is an $n \times n$ matrix. Which of the following conditions guarantee that $\lambda = 4$ is an eigenvalue of A? Clearly circle all that apply.

(i) The equation $(A - 4I)x = 0$ has infinitely many solutions.

(ii) There is a nonzero vector x in $\mathbf{R}^n$ so that the set $\{x, Ax\}$ is linearly dependent.

(iii) There is a non-trivial solution to the equation $Ax = 4x$.

(iv) $\mathrm{Nul}(A - 4I) = \{0\}$.

c) (3 points) Suppose A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(\lambda + 1)^2.$$

Which of the following statements are true? Clearly circle all that apply.

(i) The eigenvalues of A are -1 and 0.

(ii) A cannot be diagonalizable.

(iii) The null space of A must be 1-dimensional.

Problem 4.

Parts (a), (b), and (c) are unrelated. Briefly show your work on (c).

a) Let A be the 3×3 matrix for projection onto the xy-plane in $\mathbf{R}^3$.
(i) (2 points) What are the eigenvalues of A?

(ii) (1 point) Is A diagonalizable? YES NO

b) (4 points) Let A be the 2×2 matrix that reflects vectors across the line $y = x$. Fill in the blanks below.

One eigenvalue of A is $\lambda_1 =$ ______ and an eigenvector for λ_1 is $v_1 = \begin{pmatrix} \\ \\ \end{pmatrix}$.

The other eigenvalue of A is $\lambda_2 =$ ______ and an eigenvector for λ_2 is $v_2 = \begin{pmatrix} \\ \\ \end{pmatrix}$.

c) (3 points) Suppose A is a 2×2 positive stochastic matrix with the property that as n gets very large, $A^n \begin{pmatrix} 90 \\ 70 \end{pmatrix}$ approaches $\begin{pmatrix} 120 \\ 40 \end{pmatrix}$.
What is the steady-state vector w for A?

Problem 5.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{pmatrix}$.

a) (2 points) Find the eigenvalues of A. You do not need to show your work on this part.

b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) (3 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Problem 6.

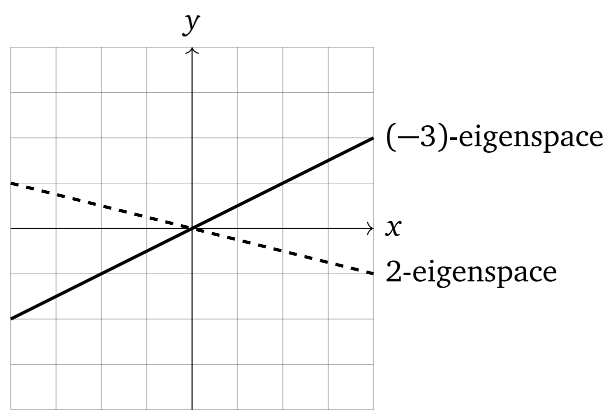
Free response. Fully simplify your answers. Parts (a) and (b) are unrelated. Show your work! A correct answer without sufficient work may receive little or no credit.

a) Let $A = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix}$.

(i) (3 points) Find the complex eigenvalues of A.

(ii) (3 pts) For the eigenvalue with positive imaginary part, find an eigenvector v.

b) (4 points) Let A be the 2×2 matrix whose (-3)-eigenspace is the **solid** line below and whose 2-eigenspace is the **dashed** line below.



Find $A\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Enter your answer here: $A\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Problem 7.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) Awesome Coffee and Bunk Coffee compete for a market of 60 customers who drink coffee every day. Today, Awesome Coffee has 50 daily customers and Bunk Coffee has 10 daily customers.
Each day:

- 60% of Awesome Coffee customers keep drinking Awesome Coffee, while 40% switch to Bunk Coffee.
- 80% of Bunk Coffee customers keep drinking Bunk Coffee, while 20% switch to Awesome Coffee.

(i) (3 points) Write a positive stochastic matrix A **and** a vector x so that Ax will give the number of customers for Awesome Coffee and Bunk Coffee (in that order) tomorrow. You do not need to compute Ax.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad\qquad x = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

(ii) (4 points) In the long term, approximately how many daily customers will Awesome Coffee have?

To receive full credit, you needed to answer the question that was directly asked. This means that your answer must be a **number**, not a vector. If you just wrote $\begin{pmatrix} 20 \\ 40 \end{pmatrix}$ and did not interpret that vector to answer the question, you did not receive full credit.

b) (3 points) Find $\det \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 6 & 0 & 1 \end{pmatrix}$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.