MATH 1553, SPRING 2022
MIDTERM 3

Name		GT ID	

Circle your lecture below.

Jankowski, lecture A (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Yu, lecture G (12:30-1:20 PM)

Leykin, lecture I (2:00-2:50 PM) Leykin, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Unless specified otherwise, the entries of all matrices are real numbers.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:15 PM on Wednesday, April 13.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) If A is a 4×4 matrix, then $\det(-A) = -\det(A)$.

TRUE FALSE

b) Suppose A is an $n \times n$ matrix and B is its RREF. If $\det(B) = 0$, then $\det(A) = 0$.

TRUE FALSE

c) If u and v are eigenvectors of an $n \times n$ matrix A, then $u + v$ must also be an eigenvector of A.

TRUE FALSE

d) Suppose A is a 2×2 matrix. If $A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$, then $A - 3I$ is not invertible.

TRUE FALSE

e) Suppose A is a 6×6 matrix with exactly two eigenvalues $\lambda = 2$ and $\lambda = -5$. If $\dim(\mathrm{Nul}(A - 2I)) = 5$, then A must be diagonalizable.

TRUE FALSE

Problem 2.

Short answer and multiple choice. You do not need to show your work, and there is no partial credit.

a) Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = -2$. Find the determinant of $(A^T B^{-1})^2$.

(i) -4 (ii) 2 (iii) -2 (iv) $1/4$

(v) 4 (vi) $1/16$ (vii) 16 (viii) none of these

b) Match each of the following linear transformations $\mathbf{R}^2 \to \mathbf{R}^2$ with the eigenvalues of its standard matrix. For each roman numeral (i) through (iv), write the corresponding set of eigenvalues (one of the choices (1) through (5)). It is possible that the same answer choice will be used more than once.

(i) Rotation countercl. by $\pi/3$ radians
(ii) Rotation clockwise by π radians
(iii) Reflection across the line $y = x$
(iv) Projection onto the y-axis

(1) $\lambda = 0$ and $\lambda = 1$
(2) $\lambda = 1$ and $\lambda = -1$
(3) $\lambda = 1$ only
(4) $\lambda = -1$ only
(5) No real eigenvalues

Write your answer for (i) here: ________

Write your answer for (ii) here: ________

Write your answer for (iii) here: ________

Write your answer for (iv) here: ________

c) Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1}$. Which of the following statements are true? Select all that apply.

(i) $A^n \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ approaches the zero vector as $n \to \infty$.

(ii) $A\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(iii) Repeated multiplication by A pushes vectors towards the $(-1/2)$-eigenspace.

(iv) A is invertible.

Problem 3.

Short answer and multiple choice. Show your work on part (a). You do not need to show your work on parts (b) and (c), and there is no partial credit on (b) and (c).

a) (3 points) Find the area of the triangle with vertices $(0,0)$, $(1,4)$, and $(5,2)$. Show your work on this part of the problem.

b) (4 points) Suppose A is a square matrix and its characteristic polynomial is

$$\det(A-\lambda I) = -\lambda^3 - 2\lambda^2 + 1.$$

Which of the following statements must be true? Circle all that apply.

(i) A is a 3×3 matrix.

(ii) $\det(A) = -1$.

(iii) The matrix transformation $T(v) = Av$ is onto.

(iv) $A - I$ is invertible.

c) (3 points) Suppose A is a 2×2 positive stochastic matrix. Which of the following statements are true? Circle all that apply.

(i) The equation $(A-I)x = 0$ has exactly one solution.

(ii) The 1-eigenspace of A is a line.

(iii) All of the eigenvalues of A are real.

Problem 4.

Short answer and multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

a) (2 points) Write the value of c so that $\lambda = 1$ is an eigenvalue of the matrix $\begin{pmatrix} 2 & 4 \\ 3 & c \end{pmatrix}$.

Fill-in the blank: $c =$ ________.

b) (3 points) Let A be a 4×4 matrix whose entries are real numbers. Which polynomials below are possible for the characteristic polynomial of A? Select all that apply.

(i) $(\lambda-1)^2(\lambda-2)^2$

(ii) $(\lambda-i)^2(\lambda-1)^2$

(iii) $(\lambda-1)^3(\lambda+1)(\lambda-2)$

c) (3 points) In each case, determine whether the matrix is diagonalizable. Clearly circle your answers.

(i) $A = \begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$ Diagonalizable Not Diagonalizable

(ii) $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ Diagonalizable Not Diagonalizable

(iii) $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Diagonalizable Not Diagonalizable

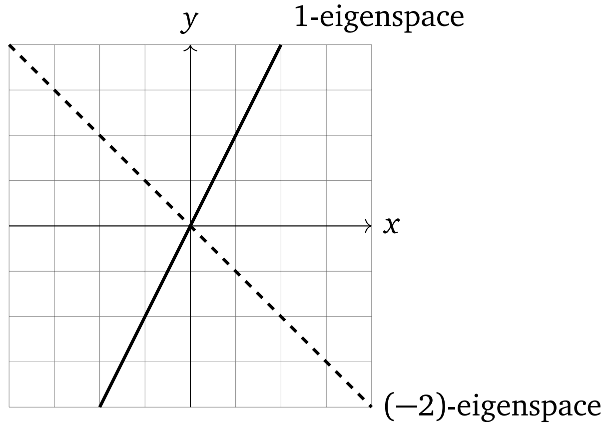
d) (2 points) Consider the positive stochastic matrix $A = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix}$. The 1-eigenspace of A is spanned by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 17 \\ 4 \end{pmatrix}$ approach as n gets very large?

(i) $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ (ii) $\begin{pmatrix} 17 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 51 \\ 16 \end{pmatrix}$ (iv) $\begin{pmatrix} 9/21 \\ 12/21 \end{pmatrix}$ (v) $\begin{pmatrix} 51/67 \\ 16/67 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Problem 5.

Free response! Show your work. Parts (a) and (b) are unrelated.

a) (4 points) Let A be a 2×2 matrix whose 1-eigenspace is the **solid** line below and whose (-2)-eigenspace is the **dashed** line below.



Find $A\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

b) (6 points) Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$. Find the eigenvalues of A. For the eigenvalue whose imaginary part is *positive*, find one corresponding eigenvector.

Problem 6.

Free response. Show your work!

Consider $A = \begin{pmatrix} 1 & 5 & 10 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$.

a) (2 points) Write the eigenvalues of the matrix A.

b) (5 points) For each eigenvalue, find a basis for the corresponding eigenspace.

c) (3 points) Is A diagonalizable? If so, write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

Problem 7.

Free response. Show your work! Parts (a) and (b) are unrelated.

a) (3 points) Find all values of k so that the matrix below is invertible.

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 3 & 4 & 1 & 2 \\ 0 & 1 & k & 1 \end{pmatrix}.$$

b) Amara's coffee shop competes with Ben's cafe for customers. Currently in 2022, Amara has 210 customers and Ben has 140 customers.

Each year:

- 70% of Amara's customers stay with Amara, while 30% of Amara's customers switch to Ben.
- 60% of Ben's customers stay with Ben, while 40% of Ben's customers switch to Amara.

Answer the following questions.

(i) (4 points) Write a positive stochastic matrix A and a vector x so that Ax is the vector that gives the number of customers for Amara and Ben (in that order) in 2023. Do not carry out the multiplication! Just write A and x.

(ii) (3 points) Find the steady-state vector w for the matrix A.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.