MATH 1553, JANKOWSKI
MIDTERM 3, SPRING 2019

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

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Problem 1. [10 points]

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer.

a) **T** **F** If A is an $n \times n$ matrix, then the determinant of A is the same as the determinant of the RREF of A.

b) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)(-1-\lambda)^2,$$

then A must be invertible.

c) **T** **F** Suppose A is an $n \times n$ matrix and λ is an eigenvalue of A. If v and w are two different eigenvectors of A corresponding to the eigenvalue λ, then $v - w$ is an eigenvector of A.

d) **T** **F** If A and B are 3×3 matrices that have the same eigenvalues and the same algebraic multiplicity for each eigenvalue, then $A = B$.

e) **T** **F** If A is a 4×4 matrix, then $\det(-A) = \det(A)$.

Solution.

a) False. If this were true then every invertible matrix would have determinant 1.

b) True. From the characteristic polynomial we see that 0 is not an eigenvalue of A.

c) True: $A(v-w) = Av - Aw = \lambda v - \lambda w = \lambda(v-w)$, and $v - w \neq 0$ since the statement says $v \neq w$.

d) False. For example, $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and I.

e) True: each row is multiplied by -1, so $\det(-A) = (-1)^4 \det(A) = \det(A)$.

Extra space for scratch work on problem 1

Problem 2.

[12 points]

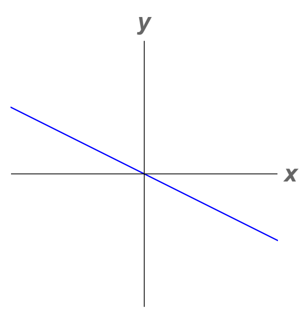
You do not need to show your work or justify your answers. Each part is worth 2 points except (e), which is worth 4 points.

a) Complete the following definition (be mathematically precise!):
Suppose A is an $n \times n$ matrix and λ is a real number. We say λ is an *eigenvalue* of A if...

b) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 3$. Find $\det\begin{pmatrix} -4a+d & -4b+e & -4c+f \\ a & b & c \\ g & h & i \end{pmatrix}$.

c) Write a 2×2 matrix which is neither diagonalizable nor invertible.

d) Let A be the matrix which implements reflection in $\mathbf{R}^2$ across the line $y = -x/2$. In the graph below, clearly draw one eigenvector in each eigenspace of A. (you don't need to write the eigenvalues of A)



e) Suppose A is an $n \times n$ matrix and $\det(A) = 0$.
Which of the following statements must be true? Circle all that apply.

(i) $\dim(\mathrm{Nul}(A)) \geq 1$.

(ii) The equation $Ax = 0$ has only the trivial solution $x = 0$.

(iii) $\lambda = 0$ is an eigenvalue of A.

(iv) The equation $Ax = b$ must be inconsistent for at least one b in $\mathbf{R}^n$.

Solution.

a) ... the equation $Ax = \lambda x$ has a non-trivial solution.

b) The matrix is obtained from the original by swapping the first two rows and then doing a row replacement, so the determinant is $(-1)(3) = -3$.

c) Many examples possible. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

d) Any nonzero along the line $y = -x/2$ is an eigenvector (for $\lambda = 1$) since it is fixed by A. Any nonzero vector perpendicular to the line $y = -x/2$ (so, on the line $y = 2x$) is an eigenvalue for $\lambda = -1$.

e) (i), (iii), and (iv) are true, but (ii) is not. Since $\det(A) = 0$ we know A is not invertible, which means it has more than just the zero vector in its nullspace, and $\lambda = 0$ is an eigenvalue, and the transformation $T(x) = Ax$ is not onto.

Extra space for work on problem 2

Problem 3. [9 points]

Parts (a) and (b) are unrelated.

a) Let $A = \begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$. Find the complex eigenvalues of A. For the eigenvalue with positive imaginary part, find one corresponding eigenvector.

Calculations show the characteristic equation of A is $\lambda^2 - 4\lambda + 5 = 0$.

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i.$$

For $\lambda = 2+i$ we have

$$\big(A - (2+i) \mid 0\big) = \left(\begin{array}{cc|c} 5-(2+i) & 5 & 0 \\ & * & 0 \end{array}\right) = \left(\begin{array}{cc|c} 3-i & 5 & 0 \\ & * & 0 \end{array}\right).$$

Using the familiar quick trick from class we get $v = \begin{pmatrix} -5 \\ 3-i \end{pmatrix}$. Alternative methods give equivalent correct answers like $\begin{pmatrix} -3-i \\ 2 \end{pmatrix}$ and $\begin{pmatrix} \frac{-3-i}{2} \\ 1 \end{pmatrix}$, which are scalar multiples of the v we wrote.

b) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & c & c & 1 \\ 3 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find all values of c so that $\det(A) = 4$.

Cofactor expansion along the 3rd column gives

$$\det(A) = c(-1)^5 \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 1 & 0 & 1 \end{pmatrix} = -c(1(0) - 2(-1) + 3(0)) = -2c.$$

Thus $4 = -2c$, so $c = -2$.

Extra space for work on problem 3

Problem 4.

[10 points]

Parts (a) and (b) are unrelated.

a) Let $A = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. The eigenvalues of A are $\lambda = 1$ and $\lambda = 3$.

(i) Find a basis for each eigenspace of A.

$$\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $x_1 = x_3$ and x_2 and x_3 are free. The 1-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$$\left(A - 3I \mid 0\right) = \left(\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so $x_1 = x_2$ and x_3 is free. The 3-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(ii) A is diagonalizable. Write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

C consists of eigenvectors and D consists of the corresponding eigenvalues in the appropriate order.

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

b) Find a 2×2 matrix B whose 1-eigenspace is the line $y = -x$ and whose (-2)-eigenspace is the y-axis. Simplify your answer completely.

The 1-eigenspace is Span$\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ and the (-2)-eigenspace is Span$\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

$$B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & -2 \end{pmatrix}$$

Extra space for work on problem 4

Problem 5.

[9 points]

Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day.
Today, Courage has 80 customers and Dexter has 130 customers. Each day:

70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.

40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) Find the steady-state vector for A.

$$\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.3 & 0.6 & 0 \\ 0.3 & -0.6 & 0 \end{array}\right) \xrightarrow[R_1 = R_1/(-0.3)]{R_2 = R_2 + R_1} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $x_1 = 2x_2$ and x_2 is free. A 1-eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the steady state vector is $w = \dfrac{1}{2+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

c) Use your answer from (b) to determine the following: in the long run, roughly how many daily customers will Courage Soda have?

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 cusomters.

Extra space for work on problem 5