MATH 1553, EXAM 3
FALL 2023

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A, 8:25-9:15 AM)   Kafer (B, 8:25-9:15 AM)   Irvine (C, 9:30-10:20)

Kafer (D, 9:30-10:20 AM)   He (G, 12:30-1:20 PM)   Goldsztein (H, 12:30-1:20)

Goldsztein (I, 2:00-2:50 PM)   Neto (L, 3:30-4:20 PM)

Yu (M, 3:30-4:20 PM)   Ostrovskii, (N, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Unless stated otherwise, the entries in all matrices are **real** numbers.
- As always, RREF means “reduced row echelon form.”
- The “zero vector” in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, November 15.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

In each statement, A is a matrix whose entries are **real numbers**.

a) Suppose A is a 3×3 matrix and there is some b in $\mathbf{R}^3$ so that the equation $Ax = b$ has exactly one solution. Then A must be invertible.

TRUE FALSE

b) If A is an $n \times n$ matrix and $\det(A) = 0$, then $\lambda = 0$ must be an eigenvalue of A.

TRUE FALSE

c) There is a 3×3 real matrix A whose eigenvalues are -1, 3, and $2+i$.

TRUE FALSE

d) Suppose A is a 4×4 matrix and its eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \lambda_3 = 5, \quad \lambda_4 = 7.$$

Then A must be diagonalizable.

TRUE FALSE

e) If A is a 5×5 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(\lambda+2)(\lambda-4)^3,$$

then the null space of A must be a line.

TRUE FALSE

Problem 2.

Parts (a) through (d) are unrelated. You do not need to show your work on this page.

a) (2 points) Let $A = \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$. Find A^{-1}. Clearly circle your answer below.

(i) $\frac{1}{10}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ (ii) $\frac{1}{10}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ (iii) $-\frac{1}{2}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

(iv) $\frac{1}{10}\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$ (v) $\frac{1}{10}\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ (vi) $-\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$

b) (3 points) Suppose A is an invertible matrix whose inverse is given by

$$A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(i) Suppose b is a vector in $\mathbf{R}^3$. How many solutions will the equation $Ax = b$ have? Circle your answer below.

None Exactly one Infinitely many solutions Not enough info to tell

(ii) Which **one** of the vectors below is a solution to $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$? Circle your answer.

$x = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ $x = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$ $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $x = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 0 \\ -1/3 \end{pmatrix}$

c) (2 pts) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}$.

Clearly circle your answer below.

(i) 1 (ii) −1 (iii) 2 (iv) −2

(v) 4 (vi) −4 (vii) 8 (viii) −8.

d) (3 points) Suppose A and B are 2×2 matrices satisfying

$$\det(A) = 6, \qquad \det(B) = -3.$$

Which of the following statements must be true? Clearly circle all that apply.

(i) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) $\det(3B^{-1}) = -1$.

(iii) $A - 6I$ is not invertible.

Problem 3.

Parts (a), (b), (c), and (d) are unrelated. There is no work required on this page.

a) Suppose A is an $n \times n$ matrix. Which **one** of the following statements is **not** correct?

(i) An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.

(ii) An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.

(iii) An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.

(iv) An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

b) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = 7x$, and let A be the standard matrix for T, so $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.
In the blank below, write one eigenvector v in the (-1)-eigenspace of A.

$$v = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

c) (2 points) Let $A = \begin{pmatrix} -1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Which **one** of the following describes the 1-eigenspace of A?

(i) $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ (ii) Span $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ (iii) Span $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$ (iv) Span $\left\{\begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}\right\}$

(v) Span $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$ (vi) Span $\left\{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}\right\}$ (vii) All of $\mathbf{R}^3$

d) (4 points) Let $A = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$. Which of the following are true? Clearly circle all that apply.

(i) The eigenvalues of A are $1/2$ and 1.

(ii) For each vector x in $\mathbf{R}^2$, it is the case that $A^n x$ approaches $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ as n becomes large.

(iii) $\mathrm{Nul}(A - I) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$

(iv) $A^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Problem 4.

You do not need to show your work on this problem. Parts (a), (b), and (c) are unrelated.

a) (2 points) Find all values of c so that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{pmatrix} 4 & -3 \\ 4 & c \end{pmatrix}$. Clearly circle your answer below.

(i) $c = -3$ only (ii) $c = -4$ only (iii) $c = 4$ only (iv) $c = -6$ only

(v) All c except -4 (vi) All c except -6 (vii) All c except 6.

b) (4 points) The characteristic polynomial of some 7×7 matrix A is

$$\det(A - \lambda I) = (5 - \lambda)(1 - \lambda)^2(\pi - \lambda)^4.$$

For this particular matrix A, some information is given below.

Eigenvalue	5		
Algebraic multiplicity			
Geometric multiplicity		2	3

(i) In the table above, write its missing entries.

(ii) Is A diagonalizable? Clearly circle your answer below.

YES NO NOT ENOUGH INFORMATION

c) (4 points) Suppose A is a 3×3 matrix. Which of the following statements are true? Clearly circle all that apply.

(i) If B is a 3×3 matrix that has the same reduced row echelon form as A, then the eigenvalues of B are the same as the eigenvalues of A.

(ii) If $\lambda = 3$ is an eigenvalue of A, then the equation $Ax = 3x$ must have infinitely many solutions.

(iii) If the equation $(A - 2I)x = 0$ has only the trivial solution, then 2 is not an eigenvalue of A.

(iv) It is impossible for A to have 4 different eigenvalues.

Problem 5.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} -4 & -8 & 12 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$.

a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.

b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) (3 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

Problem 6.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

a) Let $A = \begin{pmatrix} 5 & -5 \\ 4 & -3 \end{pmatrix}$.

(i) (4 points) Find the complex eigenvalues of A. Fully simplify your answer.

(ii) (3 points) For the eigenvalue with *positive* imaginary part, find one corresponding eigenvector v. Enter your answer in the space below.

$$v = \begin{pmatrix} \\ \\ \end{pmatrix}$$

b) (3 points) Given that

$$\det\begin{pmatrix} 0 & -1 & 3 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 28, \quad \det\begin{pmatrix} 4 & 2 & -1 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 8, \quad \text{and } \det\begin{pmatrix} 4 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 4 & 2 \end{pmatrix} = -56,$$

compute the determinant of the 4×4 matrix W below.

$$W = \begin{pmatrix} 4 & 1 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 4 & 2 \\ -2 & -1 & -1 & 1 \end{pmatrix}$$

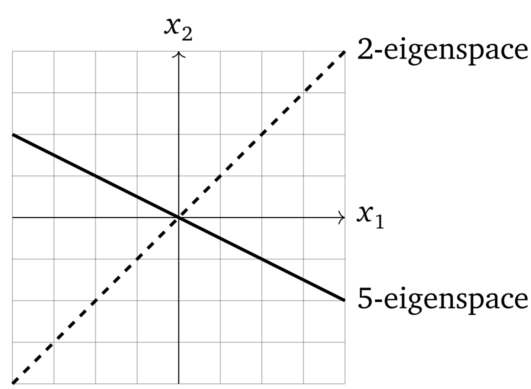
Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) (5 points) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$. Find A^{-1}. Write your answer in the space below.

$$A^{-1} = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

b) (5 points) Let A be the 2×2 matrix whose 5-eigenspace is the **solid** line below and whose 2-eigenspace is the **dashed** line below. Find $A\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.



This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.