MATH 1553, JANKOWSKI
MIDTERM 2, SPRING 2018, LECTURE C

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work unless instructed otherwise. A correct answer without appropriate work will receive little or no credit. If you cannot fit your work on the front side of the page, use the back side of the page as indicated.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[2 points for each part]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to justify your answer.

a) **T** **F** If A is an $n \times n$ matrix and $Ax = 0$ has only the trivial solution, then the equation $Ax = b$ is consistent for every b in $\mathbf{R}^n$.

b) **T** **F** If a matrix A has more rows than columns, then the linear transformation T given by $T(x) = Ax$ is not onto.

c) **T** **F** If a set S of vectors contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

d) **T** **F** There are linear transformations $T : \mathbf{R}^4 \to \mathbf{R}^3$ and $U : \mathbf{R}^3 \to \mathbf{R}^4$ so that $T \circ U$ is invertible.

e) **T** **F** If A and B are 3×3 matrices and the columns of B are linearly dependent, then the columns of AB are linearly dependent.

Solution.

a) True. A is invertible by the Inv. Mtx. Thm.

b) True. Since A can have at most one pivot in each column, and there are more rows than columns, A cannot have a pivot in every row.

c) False. For example, the set $S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\}$ in $\mathbf{R}^3$ is linearly dependent.

d) True. This is like the last problem we did in class for section 2.9. Take $U(x_1, x_2, x_3) = (x_1, x_2, x_3, 0)$ and $T(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. Then $(T \circ U)(x_1, x_2, x_3) = (x_1, x_2, x_3)$ so $T \circ U$ is invertible (in fact, it is the identity transformation!)

e) True. $Bv = 0$ for some nonzero v in $\mathbf{R}^3$, so $ABv = A0 = 0$. By the IMT, AB is not invertible since it is a square matrix with a non-trivial homogeneous solution.

Extra space for scratch work on problem 1

Problem 2. [12 points]

Parts (a) to (d) are unrelated. You do not need to justify answers in (a) or (b).

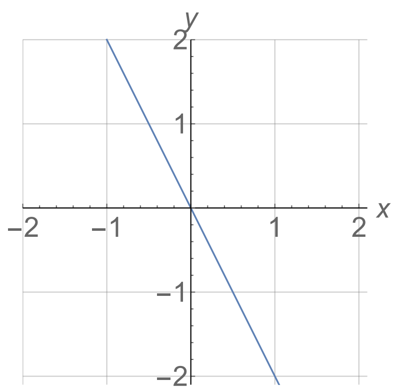
a) Write three different nonzero vectors v_1, v_2, v_3 in $\mathbf{R}^3$ so that $\{v_1, v_2, v_3\}$ is linearly dependent but v_3 is not in $\text{Span}\{v_1, v_2\}$. Clearly indicate which vector is v_3.

b) Fill in the blanks: If A is a 5×6 matrix and its column span has dimension 2, then the null space of A is a ________-dimensional subspace of $\mathbf{R}^{\square}$.

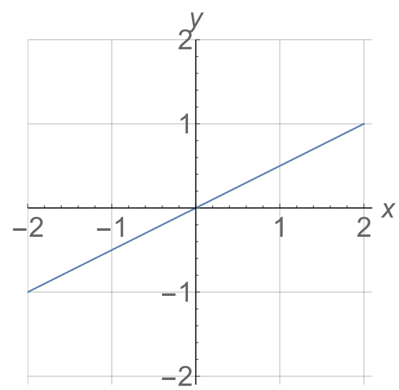
c) Find the matrix A satisfying $A^{-1}e_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $A^{-1}e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $A^{-1}e_3 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

d) Let $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Clearly draw Col($A$) and Nul($A$). Briefly show work.

Draw Col(A) here.



Draw Nul(A) here.



Solution.

a) Many examples possible. For example, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

b) Nul$A$ is a subspace of $\mathbf{R}^6$, and $\dim(\text{Col } A) + \dim(\text{Nul } A) = 6$ so $2 + \dim(\text{Nul}A) = 6$. Thus Nul A is a 4-dimensional subspace of $\mathbf{R}^6$.

c) $\left(A^{-1} \mid I\right) = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right)$

$\xrightarrow{R_1 = R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right)$. Thus, $A = (A^{-1})^{-1} = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.

d) Col(A) is the span of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; Nul($A$) is the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For each drawing, you needed to draw the full picture to receive credit.

Extra space for work on problem 2

Problem 3.

[10 points]

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of reflection about the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x \\ 3y \end{pmatrix}$.

a) Write the standard matrix A for T. Is T onto?

b) Write the standard matrix B for U. Is U one-to-one? Briefly justify your answer.

c) Circle the composition that makes sense: $T \circ U$ $\quad$ $U \circ T$

d) Compute the standard matrix for the composition you circled in part (c).

Solution.

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Yes, T is onto (no justification was required for (a)).

b) $B = \begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}$. Yes, U is one-to-one because B has a pivot in every column.

c) $U \circ T$ makes sense since it sends $R^2 \to R^2 \to R^3$.

d) $BA = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 3 & 0 \end{pmatrix}$.

Extra space for work on problem 3

Problem 4.

[10 points]

Dino McBarker has put the matrix A below in its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 \\ -7 & 14 & 3 & 2 \\ 4 & -8 & -2 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis $\mathcal{B}$ for Nul(A).

b) Is $x = \begin{pmatrix} -4 \\ 2 \\ -20 \\ 2 \end{pmatrix}$ in Nul(A)? If so, find $[x]_{\mathcal{B}}$. If not, justify why x is not in Nul(A).

c) Is $\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ in Col(A)? You do not need to justify your answer.

Solution.

a) The RREF of A shows that if $Ax = 0$ then

$$x_1 = 2x_2 - 4x_4,\ x_2 = x_2,\ x_3 = -10x_4,\ x_4 = x_4.$$

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 - 4x_4 \\ x_2 \\ -10x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -10 \\ 1 \end{pmatrix}$. Thus $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -10 \\ 1 \end{pmatrix} \right\}$.

b)

$$\left(\begin{array}{cc|c} 2 & -4 & -4 \\ 1 & 0 & 2 \\ 0 & -10 & -20 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow[R_3 = -R_3/10]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow[R_4 = R_4 - R_3]{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & -4 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - R_2]{R_2 = R_2/-4} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, x is in Nul(A), in fact $x = 2b_1 + 2b_2$, so $[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

c) Yes. $\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ is $(-1) * (\text{third column of } A)$ so it is in Col(A), no work required.

Extra space for work on problem 4

Problem 5.

[8 points]

Parts (a), (b), and (c) are unrelated.
You do not need to show any work for parts (a) and (b).

a) I. Is the set $\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}\right\}$ linearly independent? YES $\boxed{\text{NO}}$

II. If A is a 3×3 matrix, is it possible that $\mathrm{Col}(A) = \mathrm{Nul}(A)$?

YES $\boxed{\text{NO}}$

(dimensions add to 3, so they can't even have the same dimension)

b) Give a specific example of a subspace of $\mathbf{R}^3$ that contains $\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

(You may express this subspace any way you like, as long as you are clear.)

There are endless possibilities. For example, $\mathbf{R}^3$ itself.

Many students just wrote the vector $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, or a few vectors, or a 3×3 matrix. None of these are subspaces and all would result in 0/2 for this part.

c) Write a 2×3 matrix A and a 3×2 matrix B so that AB is the standard matrix for the transformation of *clockwise* rotation by 90° in $\mathbf{R}^2$.
Compute AB to demonstrate that your answer is correct.

Many possibilities. For example, $A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}$ and $B = \begin{pmatrix}0 & 1\\-1 & 0\\0 & 0\end{pmatrix}$.

$$AB = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix},$$ which is clockwise rotation by 90°.

Extra space for work on problem 5