MATH 1553, EXAM 2
SPRING 2024

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, March 6.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose v_1, v_2, and b are vectors in $\mathbf{R}^n$ and the equation

$$x_1v_1 + x_2v_2 = b$$

has exactly one solution. Then $\{v_1, v_2, b\}$ must be linearly independent.

TRUE FALSE

b) Let V be the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ of the form

$$x - 3y + z = 2.$$

Then V is a subspace of $\mathbf{R}^3$.

TRUE FALSE

c) If a vector x is in the null space of a matrix A, then $4x$ is also in the null space of A.

TRUE FALSE

d) If A is a 4×3 matrix, then the matrix transformation $T(x) = Ax$ cannot be onto.

TRUE FALSE

e) If A is a 3×3 matrix and the equation $Ax = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ has exactly one solution, then A must be invertible.

TRUE FALSE

2. Multiple choice and short answer. You do not need to show work or justify your answers. Parts (a), (b), and (c) are unrelated.

a) (3 points) Suppose v_1, v_2, v_3, and v_4 are vectors in $\mathbf{R}^4$. Which of the following must be true? Clearly circle all that apply.
(i) If $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$, then the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

(ii) Suppose that the matrix A with columns v_1 through v_4 has one pivot, and let T be the matrix transformation $T(x) = Ax$. Then the range of T is a line.

(iii) If $v_1 - v_2 - v_3 + v_4 = 0$, then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

b) (4 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \ : \ y \leq 2x \right\}$.
Answer each of the following questions.
(i) Does V contain the zero vector? YES NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? YES NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? YES NO

(iv) Is there a matrix A with the property that $\mathrm{Col}(A) = V$? YES NO

c) (3 pts) Let A be a 15×7 matrix. If $\mathrm{rank}(A) \leq 3$, which of the following are possible values for $\mathrm{nullity}(A)$? Clearly circle **all** that apply.

(i) 1

(ii) 3

(iii) 4

(iv) 6

(v) 7

(vi) 9

3. Short answer. Parts (a) through (d) are unrelated. There is no work necessary on this problem.

a) (2 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for **counterclockwise** rotation by 13°.

(i) Find the standard matrix A for the transformation T.

(I) $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ \sin(13°) & \cos(13°) \end{pmatrix}$ (II) $A = \begin{pmatrix} \cos(13°) & \sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$

(III) $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$ (IV) $A = \begin{pmatrix} -\cos(13°) & -\sin(13°) \\ \sin(13°) & -\cos(13°) \end{pmatrix}$

(ii) T is invertible. Find the standard matrix B for the transformation T^{-1}.

(I) $B = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ \sin(13°) & \cos(13°) \end{pmatrix}$ (II) $B = \begin{pmatrix} \cos(13°) & \sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$

(III) $B = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$ (IV) $B = \begin{pmatrix} -\cos(13°) & -\sin(13°) \\ \sin(13°) & -\cos(13°) \end{pmatrix}$

b) (3 points) Let A be an $m \times n$ matrix, and let T be the associated linear transformation $T(x) = Ax$. Which of the following statements are true? Clearly circle all that apply.

(i) If the columns of A are linearly independent, then $\dim(\text{range}(T)) = n$.

(ii) If the columns of A are linearly dependent, then T is onto.

(iii) If T is onto, then $\text{Col}(A) = \mathbf{R}^m$.

c) (3 points) Determine which of the following statements are true.

(i) If A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) If A is a 3×7 matrix and B is a 4×3 matrix, then the matrix transformation T given by $T(x) = BAx$ has domain $\mathbf{R}^4$ and codomain $\mathbf{R}^7$.

(iii) If A is an $n \times n$ matrix and $A^2 = 0$, then $(I - A)(I + A) = I$.

d) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first rotates vectors by 90 degrees clockwise, then reflects across the line $y = -x$. Find $T\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Clearly circle your answer below.

(i) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (v) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (vi) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. Short answer. You do not need to show your work on this page. Parts (a), (b), and (c) are unrelated.

a) (4 points) Suppose A is a 2000×30 matrix and its RREF has 20 pivots. Answer each of the following questions.

(i) Fill in the blank: the rank of A is ________

(ii) Fill in the blank: the nullity of A is ________.

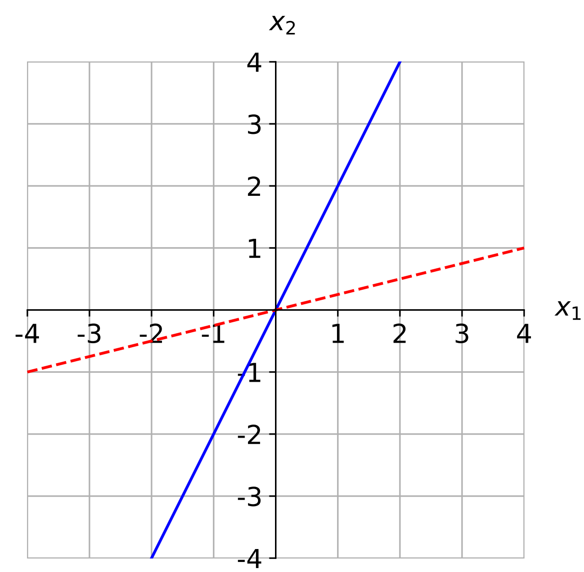
(iii) Circle the correct answer below to complete the following sentence:
Col(A) is a subspace of...

$\mathbf{R}^{10}$ $\quad$ $\mathbf{R}^{20}$ $\quad$ $\mathbf{R}^{30}$ $\quad$ $\mathbf{R}^{1980}$ $\quad$ $\mathbf{R}^{1990}$ $\quad$ $\mathbf{R}^{2000}$

(iv) Circle the correct answer below to complete the following sentence:
Nul(A) is a subspace of...

$\mathbf{R}^{10}$ $\quad$ $\mathbf{R}^{20}$ $\quad$ $\mathbf{R}^{30}$ $\quad$ $\mathbf{R}^{1980}$ $\quad$ $\mathbf{R}^{1990}$ $\quad$ $\mathbf{R}^{2000}$

b) (4 pts) Write a matrix A so that Col(A) is the solid line below and Nul(A) is the dashed line below.



c) (2 pts) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 11 \end{pmatrix}$. Find A^{-1}. Clearly circle your answer below.

(i) $A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & 2 \\ 3 & 1 \end{pmatrix}$ $\quad$ (ii) $A^{-1} = \frac{1}{17}\begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}$ $\quad$ (iii) $A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & -3 \\ -2 & 1 \end{pmatrix}$

(iv) $A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}$ $\quad$ (v) $A^{-1} = \begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}$ $\quad$ (vi) $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 11 \end{pmatrix}$

The rest of the exam is free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even it is correct.
On this page, you do not need to show your work in parts (a) or (d).

5. Consider the following matrix A below in its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 1 & -3 & 2 \\ 1 & -1 & -1 & 0 \\ 1 & 3 & -5 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col(A). You don't need to show your work on this part.

b) Write a new basis for Col(A), so that no vector in your new basis is a scalar multiple of any of the vectors in the basis you wrote in part (a). Clearly show how you obtain this new basis.

c) Find a basis for Nul(A).

d) Write **one** solution x to the equation $T(x) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$. There is no partial credit on this part, so take time to check by hand that your answer is correct.

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Free response. Show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct.

6. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation $T(x_1, x_2) = (3x_1 - x_2,\ 5x_2 - x_1, x_1 + x_2)$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects vectors across the line $y = x$.

a) (3 points) Find the standard matrix A for T.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) (2 points) Write the standard matrix B for U.

$$B = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

c) (1 point) Which **one** of the following compositions makes sense? (no work required)

$T \circ U$ $\qquad\qquad$ $U \circ T$

d) (4 points) For the composition you circled in part (c), compute the standard matrix for the transformation.

7. Free response. Show your work in (b) and (c)! Parts (a), (b), and (c) are unrelated.

a) (2 points) Suppose A is a matrix and T is the corresponding matrix transformation $T(x) = Ax$. Fill in the blanks below. You do not need to show your work on this part.

(i) T is one-to-one if A has a pivot in every ________.

(ii) T is onto if A has a pivot in every ________.

b) (4 points) Let $A = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}$. Find a nonzero matrix B so that AB is the 2×2 zero matrix. Enter your answer in the space provided below.

$$B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

c) (4 points) Find all real values of c (if there are any) so that the following set is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ c \\ -4 \end{pmatrix} \right\}$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.