MATH 1553, EXAM 2 SOLUTIONS
FALL 2023

Name		**GT ID**	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A, 8:25-9:15 AM) Kafer (B, 8:25-9:15 AM) Irvine (C, 9:30-10:20)

Kafer (D, 9:30-10:20 AM) He (G, 12:30-1:20 PM) Goldsztein (H, 12:30-1:20)

Goldsztein (I, 2:00-2:50 PM) Neto (L, 3:30-4:20 PM)

Yu (M, 3:30-4:20 PM) Ostrovskii, (N, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, October 18.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) If $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbf{R}^4$, then $\{v_1, v_2, v_3\}$ must be linearly independent.

[TRUE] FALSE

b) If A is a 30×20 matrix and $\dim(\text{Col } A) = 10$, then the null space of A is a 10-dimensional subspace of $\mathbf{R}^{20}$.

[TRUE] FALSE

c) If A is an $m \times n$ matrix and $m > n$, then the matrix transformation $T(x) = Ax$ cannot be one-to-one.

TRUE [FALSE]

d) Suppose A is a 3×2 matrix whose columns are linearly independent, and let T be the matrix transformation $T(x) = Ax$. Then

$$\left\{ T\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the range of T.

[TRUE] FALSE

e) Suppose $T : \mathbf{R}^4 \to \mathbf{R}^2$ and $U : \mathbf{R}^2 \to \mathbf{R}^3$ are matrix transformations, and let A be the standard matrix for $U \circ T$, so $(U \circ T)(x) = Ax$. Then A is a 4×3 matrix.

TRUE [FALSE]

Solution.

a) True: if $\{v_1, v_2, v_3\}$ were linearly dependent then $\{v_1, v_2, v_3, v_4\}$ would be linearly dependent which we know is not the case. This was nearly copied from #3 in the 3.1 supplement.

b) True. This is a classic Rank Theorem question. The null space of A has dimension $20-10=10$, and we know the null space of A lives in $\mathbf{R}^{20}$ because A has 20 columns.

c) False, T can be one-to-one because A can still have a pivot in every column if $m > n$. For example, when $m=3$ and $n=2$, here is such an A:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

d) True: $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$ are precisely the columns of A, which are linearly independent by assumption, therefore they form a basis for the range of T (i.e. the column space of A).

e) False: $U \circ T$ is a transformation with domain $\mathbf{R}^4$ and codomain $\mathbf{R}^3$, so A is a 3×4 matrix. This is a slighly changed #1b from the 3.4 worksheet.

Another way to do this is to note that the matrix for U is 3×2 and the matrix for T is 2×4, so the matrix for $(U \circ T)$ is a 3×2 times a 2×4, which is a 3×4 matrix.

Problem 2.

Parts (a), (b), and (c) are unrelated. There is no work required and no partial credit on this page.

a) (3 points) Consider the set $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x - y \geq 0 \right\}$.

(i) Does V contain the zero vector? $\boxed{\text{YES}}$ NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? $\boxed{\text{YES}}$ NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? YES $\boxed{\text{NO}}$

b) (3 points) Suppose $\{v_1, v_2, v_3\}$ is a set of vectors in $\mathbf{R}^n$. Which of the following statements are true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ If $\{v_1, v_2, v_3\}$ is a basis for $\mathbf{R}^n$, then $n = 3$.

(ii) If the vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ has the trivial solution, then $\{v_1, v_2, v_3\}$ must be linearly independent.

(iii) If $\{v_1, v_2, v_3\}$ is linearly dependent, then there is a nonzero number x_1, a nonzero number x_2, and a nonzero number x_3 so that $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$.

c) (4 points) Suppose A is an 11×5 matrix and T is the corresponding linear transformation given by the formula $T(x) = Ax$. Which of the following statements are true? Clearly circle all that apply.

(i) $\dim(\text{Col } A) \geq \dim(\text{Nul} A)$.

(ii) If the columns of A are linearly independent, then the range of T is $\mathbf{R}^5$.

(iii) Suppose b is a vector so that the matrix equation $Ax = b$ is consistent. Then the set of solutions to $Ax = b$ must be a subspace of $\mathbf{R}^5$.

$\boxed{\text{(iv)}}$ If the matrix equation $Ax = 0$ has infinitely many solutions, then $\text{rank}(A) \leq 4$.

Solution.

a) This is #7a from Sample Midterm 2B with "xy" replaced by "$x-y$." It is also similar to #1 from the 2.6 Webwork and #3a from Sample Midterm 2A.

(i) Yes, since $0-0 \geq 0$.

(ii) Yes: geometrically, V consists of all vectors on, and below, the line $x = y$. If you take two vectors from that region and add them together, the resulting vector will be in that region.

Alternatively, we can do algebgra instead. Suppose $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are vectors in V, so $x_1 - y_1 \geq 0$ and $x_2 - y_2 \geq 0$. Then

$$(x_1 + x_2) - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2),$$

which is the sum of two nonnegative numbers and is therefore nonnegative.

(iii) No. For example, the vector $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in V, but $-u = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ which is not in V since $-1-0 < 0$.

b) (i) Yes: the dimension of $\mathbf{R}^n$ is n, so if $\{v_1, v_2, v_3\}$ is a basis for $\mathbf{R}^n$ then $n = 3$. This is a slight modification of #4d from the 2.6 supplement.

(ii) No: that vector equation always has the trivial solution. The vectors are linearly independent precisely when the trivial solution is the **only** solution. This is #7 from the 2.5 Webwork, but with the matrix equation $Ax = 0$ replaced by the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$.

(iii) No: for linear dependence, we only need the homogeneous equation to be satisfied when **at least one** of the numbers x_1, x_2, x_3 is nonzero. For example, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly dependent, but it is not possible to satisfy the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ unless $x_3 = 0$.

c) (i) No: for example, the 11×5 zero matrix has $\dim(\operatorname{Col} A) = 0$ but $\dim(\operatorname{Nul} A) = 5$.

(ii) No: if the columns of A are linearly independent then the range of T is a 5-dimensional subspace of $\mathbf{R}^{11}$.

(iii) No: if $b \neq 0$ then the solution set to $Ax = b$ does not contain the zero vector, so it is not a subspace.

(iv) Yes: if $Ax = 0$ has infinitely many solutions, then A cannot have a pivot in every column, so it has 4 or fewer pivots.

Problem 3.

Parts (a), (b), and (c) are unrelated. You do not need to show your work.

a) (3 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation that satisfies

$$T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}6\\0\\-4\end{pmatrix}, \qquad T\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\10\end{pmatrix}.$$

Write the standard matrix A for T (in other words, the matrix A so that $T(x) = Ax$).

b) (4 points) Suppose A is a 4×3 matrix and B is a 3×2 matrix, and let T be the matrix transformation $T(x) = ABx$. Which of the following must be true? Clearly circle all that apply.

(i) The column space of AB is a subspace of $\mathbf{R}^2$.

(ii) Every vector in the null space of AB is also in the null space of A.

(iii) T has domain $\mathbf{R}^2$ and codomain $\mathbf{R}^4$.

(iv) T cannot be onto.

c) (3 points) Which of the following transformations are **linear** transformations? Clearly circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x_1, x_2, x_3) = (x_1 - x_2,\ 1 - x_1,\ x_1)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x_1, x_2) = (0,\ x_1,\ x_1)$.

(iii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1,\ x_1 x_2)$.

Solution.

a) This is a slight modification of #6a from Sample Midterm 2B.

$$T\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\0\\-2\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\end{pmatrix} = -T\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}0\\-1\\-10\end{pmatrix}, \text{ so}$$

$$A = \left(T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}3 & 0\\0 & -1\\-2 & -10\end{pmatrix}.$$

b) We took #3b from Sample Midterm 2A and modified it slightly. Conceptually, this draws heavily from #2 on the 3.4 Worksheet.

(i) No, AB is 4×2 so $\mathrm{Col}(AB)$ is a subspace of $\mathbf{R}^4$.

(ii) No, every vector in the null space of AB lives in $\mathbf{R}^2$, whereas every vector in the

null space of A lives in $\mathbf{R}^3$, so the statement is nonsense.
(iii) Yes, AB is 4×2 so the domain of T is $\mathbf{R}^2$ and the codomain is $\mathbf{R}^4$.
(iv) Yes, AB has a max of 2 pivots so it cannot have a pivot in each row and therefore T cannot be onto.

c) This is a slight modification of #4 from the 3.3 Webwork. Part (iii) was copied directly from Quiz 5.
(i) No, T is not linear since it does not send the zero vector to the zero vector, in fact $T(0,0,0) = (0,1,0)$.
(ii) Yes, T is linear, in fact $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
(iii) No, T is not linear. The "$x_1 x_2$" term gives it away. We can also show it directly. For example, for $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we have

$$T(2u) = T\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{but} \quad 2T(u) = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

so $T(2u) \neq 2T(u)$.

Problem 4.

You do not need to show your work on this problem. Parts (a), (b), (c), and (d) are unrelated. Solutions are on the next page.

a) (3 points) Suppose that A is a matrix that represents a linear transformation T from $\mathbf{R}^7$ to $\mathbf{R}^9$. In other words, T is the transformation given by the formula $T(x) = Ax$.

(i) How many rows does the matrix A have? Enter your answer here: 9

(ii) Suppose the reduced row echelon form of the matrix A contains 3 pivots. Apply the Rank Theorem to A to fill in the following blanks with numbers.

$$\dim(\text{Col } A) = \underline{\ 3\ } \qquad \dim(\text{Nul } A) = \underline{\ 4\ }.$$

b) (2 points) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation. Which **one** of the following is a definition that T is onto?

(i) For each x in $\mathbf{R}^n$, there is a vector y in $\mathbf{R}^m$ so that $T(x) = y$.

(ii) For each x in $\mathbf{R}^n$, there is at least one vector y in $\mathbf{R}^m$ so that $T(x) = y$.

(iii) For each y in $\mathbf{R}^m$, there is at least one vector x in $\mathbf{R}^n$ so that $T(x) = y$.

c) (3 points) Let V be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form

$$\begin{pmatrix} -4x_4 \\ x_2 \\ x_2 + 6x_4 \\ x_4 \end{pmatrix}.$$

Write a basis for V.

Many answers possible, for example $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \end{pmatrix} \right\}$.

d) (2 points) Which of the following linear transformations are one-to-one? Clearly circle all that apply.

(i) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors counterclockwise by $15°$.

(ii) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} x$.

Solution.

a) (i) T has domain $\mathbf{R}^7$ and codomain $\mathbf{R}^9$, so A is 9×7, therefore A has 9 rows.
(ii) A has 3 pivots, so $\dim(\text{Col } A) = 3$. By the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 7, \quad \text{so} \quad \dim(\text{Nul } A) = 4.$$

b) Statement (iii) says that T is onto.
Statement (i) just says T is a transformation. Statement (ii) is a slight modification of the definition of transformation, and in fact if any x were to have more than one y so that $T(x) = y$, then T would fail to be a transformation in the first place.

c) This was taken directly #3 from the 2.6 Webwork.

We see V is the set of all vectors of the form $x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \end{pmatrix}$, so one basis is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \end{pmatrix} \right\}.$$

Other answers are possible.

d) Parts (i) and (ii) are slight modifications of #2a and #2d in the 3.2-3.3 worksheet, and are also similar to #4a of Sample Midterm 2A.
(i) Yes, T is one-to-one. In fact, it is both one-to-one and onto!
(ii) No, T is not one-to-one. One step of row-reduction shows that A only has 2 pivots, so A has a column without a pivot.

Problem 5.

For this problem, consider the matrix A and its reduced row echelon form given below.

$$A = \begin{pmatrix} 1 & 7 & 0 & -4 \\ -1 & -7 & 1 & 7 \\ 2 & 14 & 1 & -5 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 7 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Parts (a) through (c) in this problem are #5a through #5c of Sample Midterm 2A with slightly changed numbers.

a) (2 points) Write a basis for Col A. Briefly justify your answer.

Solution: The first and third columns of A are pivot columns, so they form a basis for Col A. In reality, **any choice of two columns** of A *except for the first two* will be linearly independent and thus a basis for Col A.

$$\text{One possible answer: } \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

b) (4 points) Find a basis for Nul A.

Solution: The RREF of $(A|0)$ gives homogeneous solution set $x_1 + 7x_2 - 4x_4 = 0$ and $x_3 = -3x_4$ where x_2 and x_4 are free. Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -7x_2 + 4x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}. \quad \text{Basis for Nul A} : \left\{ \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

c) (2 points) Write one vector x that is not the zero vector and is in the null space of A. Briefly justify your answer.

Any nonzero linear combination of $\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$ is correct. For example,

$$\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -3 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \quad \text{etc.}$$

d) (2 points) Let T be the matrix transformation $T(x) = Ax$. Circle the correct answers below. You do not need to show your work on this part.

(i) The range of T is:

a point a line **[a plane]** all of $\mathbf{R}^3$ all of $\mathbf{R}^4$

(ii) The range of T is a subspace of:

$\mathbf{R}$ $\mathbf{R}^2$ **[$\mathbf{R}^3$]** $\mathbf{R}^4$

Problem 6.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation by $90°$ counterclockwise.
Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = x$.
Let $V : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation

$$V(x_1, x_2, x_3) = (x_1 - 5x_2,\ 3x_1 - 4x_3).$$

Parts (a), (b), and (d) in this problem were taken from #7 in the 3.4 Webwork.

a) (2 points) Write the standard matrix A for T.
(do *not* leave your answer in terms of sine and cosine; simplify it completely)

Solution: $A = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

b) (2 points) Write the standard matrix B for U.

Solution: $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$, so

$$B = \left(U\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad U\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

c) (3 points) Find the standard matrix C for V.

Solution: $C = \left(V\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad V\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad V\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$

d) (3 points) Find the standard matrix D for the transformation $W : \mathbf{R}^2 \to \mathbf{R}^2$ that first reflects vectors in $\mathbf{R}^2$ across the line $y = x$, then rotates vectors by $90°$ counterclockwise.

Solution: Since we are doing U first and then T, our composition is $T \circ U$, so the matrix is $D = AB$.

$$D = AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Alternatively, we could just follow the steps for $W\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $W\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$W\begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$W\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a), (b), and (c) are unrelated.

a) (4 points) Let $A = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}$.

Find all values of a and b so that $A^2 = A$.

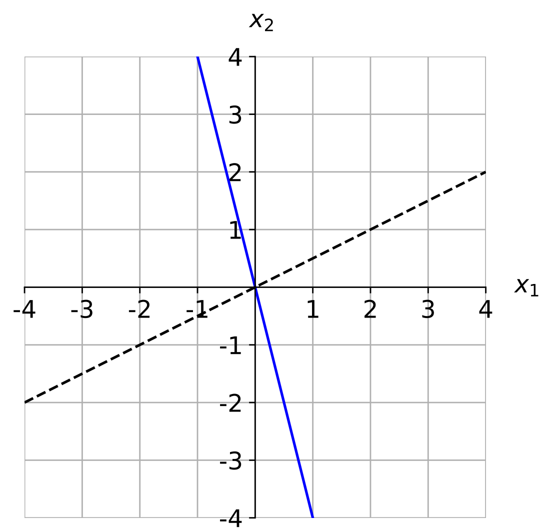
This problem was taken from #6 in the 3.4 Webwork.

Solution: $\boxed{a = 12}$ and $\boxed{b = -3}$.

$$A^2 = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}\begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix} = \begin{pmatrix} 16-a & 4a+ab \\ -4-b & -a+b^2 \end{pmatrix}, \qquad A = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}.$$

Setting $A^2 = A$ gives $16 - a = 4$ in the "11" entry, so $a = 12$. Now, in the "21" entry we have $-4 - b = -1$, so $b = -3$. One can also check that the other two entries are satisfied: $4a + ab = a$ since $48 - 3(12) = 12$, and $-a + b^2 = b$ since $-12 + (-3)^2 = -3$.

b) (4 points) Write a single matrix A with the property that Col(A) is the solid line graphed below and Nul(A) is the dotted line graphed below.



This is #7b from Sample Midterm 2A and #3 from the 2.5-3.1 Worksheet with slightly different lines.

Solution: We need $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right\}$ and $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$, so each column of A must be a multiple of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ must equal $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (in other words, the homogeneous system $Ax = 0$ will have parametric form $x_1 = 2x_2$ where x_2 is free, thus $x_1 - 2x_2 = 0$).

A correct answer A must have each column a multiple of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and the second column must be -2 times the first. For example,

$$A = \begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -4 \\ -8 & 16 \end{pmatrix}, \quad A = \begin{pmatrix} 1/2 & -1 \\ -2 & 4 \end{pmatrix}, \quad \text{etc.}$$

c) (2 points) Give one specific example of a subspace V of $\mathbf{R}^3$ that contains the vector $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Briefly justify your answer.

Solution: Note that the single vector $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ by itself is NOT a subspace of $\mathbf{R}^3$ since it does not contain the zero vector, is not closed under addition, and is not closed under scalar multiplication!

Many answers are possible. For example, $V = \text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a subspace that contains $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Another such subspace is the xy-plane of $\mathbf{R}^3$, or in other words

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Another answer is $V = \mathbf{R}^3$, since $\mathbf{R}^3$ is a subspace of itself and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is certainly in $\mathbf{R}^3$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.