MATH 1553, C. JANKOWSKI
MIDTERM 2

Name		**Section**	

Please **read all instructions** carefully before beginning.

- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work unless specified otherwise. A correct answer without appropriate work may be given little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Scoring Page

Please do not write on this page.

1	2	3	4	5	Total

Problem 1.

[Parts a) through f) are worth 2 points each]

a) Complete the following definition (be mathematically precise!):
A set of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbf{R}^n$ is *linearly independent* if...

b) Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$. If A is invertible, find A^{-1}. If A is not invertible, justify why.

The remaining problems are true or false. Answer true if the statement is *always* true. Otherwise, answer false. You do not need to justify your answer.

c) **T** **F** If A is an $n \times n$ matrix and the columns of A span $\mathbf{R}^n$, then $Ax = 0$ has only the trivial solution.

d) **T** **F** If A is a 6×7 matrix and the null space of A has dimension 4, then the column space of A is a 2-plane.

e) **T** **F** If A is an $n \times n$ matrix and $Ax = b$ has exactly one solution for some b in $\mathbf{R}^n$, then A is invertible.

f) **T** **F** If A is an $m \times n$ matrix and $m > n$, then the linear transformation $T(x) = Ax$ cannot be one-to-one.

Solution.

a) the equation $x_1 v_1 + \cdots + x_p v_p = 0$ has only the trivial solution $x_1 = \cdots = x_p = 0$.

b) $\det(A) = 2 - (-2) = 4$, so A is invertible; $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

c) True. A is invertible by the Invertible Matrix Theorem, so $Ax = 0$ has only the trivial solution.

d) False. By the Rank Theorem, $\dim(\text{Col } A) + \dim(\text{Nul } A) = 7$, so $\dim(\text{Col } A) = 3$.

e) True. If $Ax = b$ has exactly one solution for some b, then $Ax = 0$ has exactly one solution (since the sol. set for $Ax = b$ is a translate of the sol. set for $Ax = 0$), hence A is invertible.

f) False. $T : \mathbf{R}^n \to \mathbf{R}^m$ can be one to one. For example, $T(a) = (a, 0)$.

Problem 2. [10 points]

Parts (a), (b), and (c) are unrelated.

a) Let $V = \text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Fill in the blank: the dimension of V is ________.

b) Let $W = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y - z = 0\right\}$.

Is W a subspace of $\mathbf{R}^3$? (no justification required)

YES NO

c) The famous philologist is obsessed with the set of vectors

$$\left\{\begin{pmatrix}-1\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\h\\-7\end{pmatrix}\right\}$$

where h is some real number.
Find all values of h that make the set linearly dependent.

Solution.

a) A basis for V is $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, so $\dim V = 2$.

b) W is a subspace of $\mathbf{R}^3$. In fact, it is Nul A for the matrix $A = \begin{pmatrix}1 & -1 & -1\end{pmatrix}$.

c)

$$\begin{pmatrix}-1 & 1 & 1\\3 & 1 & h\\-1 & -1 & -7\end{pmatrix} \xrightarrow[R_3=R_3-R_1]{R_2=R_2+3R_1} \begin{pmatrix}-1 & 1 & 1\\0 & 4 & h+3\\0 & -2 & -8\end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix}-1 & 1 & 1\\0 & -2 & -8\\0 & 4 & h+3\end{pmatrix} \xrightarrow{R_3=R_3+2R_2} \begin{pmatrix}-1 & 1 & 1\\0 & -2 & -8\\0 & 0 & h-13\end{pmatrix}.$$

The vectors are linearly dependent if and only if the matrix has fewer than 3 pivots. The matrix will have three pivots unless $h - 13 = 0$, which is when $\boxed{h = 13}$.

Problem 3.

[11 points]

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 - x_2 \\ 2x_1 \end{pmatrix},$$

and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be reflection about the line $y = x$.

a) Write the standard matrix A for T and the standard matrix B for U.

b) Is U one-to-one? Briefly justify your answer.

c) Find the standard matrix for $U \circ T$.

d) Is the transformation $U \circ T$ onto? Briefly justify your answer.

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) U is one-to-one, since B has a pivot in every column. Alternatively, if $U(x, y)$ is the zero vector then $(y, x) = (0, 0)$ so $x = y = 0$, which shows U is one-to-one.

c) The matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

d) $U \circ T$ is onto, since BA has a pivot in every row.

Problem 4. [10 points]

Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \\ -2 & 4 & -8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for Nul A.

b) Find a basis $\mathcal{B}$ for Col A.

c) Let $x = \begin{pmatrix} -2 \\ -1 \\ -1 \\ 4 \end{pmatrix}$. Is x in Col A?

If your answer is no, justify why x is not in Col A.
If your answer is yes, find $[x]_{\mathcal{B}}$.

Solution.

a) From the RREF of $\left(A \mid 0\right)$ we see that $Ax = 0$ when $x_1 = 2x_2$, $x_2 = x_2$, and $x_3 = 0$. In parametric vector form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \text{so a basis for Nul } A \text{ is } \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

b) The RREF of A shows that the first and third columns are pivot columns.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ -8 \end{pmatrix} \right\}.$$

c) We attempt to solve $x = c_1 b_1 + c_2 b_2$ for some scalars c_1 and c_2.

$$\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 1 & 3 & -1 \\ -2 & -8 & 4 \end{array}\right) \xrightarrow[R_4 = R_4 + 2R_1]{R_3 = R_3 - R_1} \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 = R_1 - 4R_2]{R_3 = R_3 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So $c_1 = 2$ and $c_2 = -1$, and $x = 2b_1 - b_2$.

$$[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Problem 5.

[7 points]

Parts (a) and (b) are unrelated.

a) Suppose that a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $T\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find $T\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

b) Write a single matrix A that satisfies both of the following two properties:
- Col A is a subspace of $\mathbf{R}^4$, and
- Nul A is the line $y = 10x$ in $\mathbf{R}^2$.

Solution.

a) $\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so by linearity of T,

$$T\begin{pmatrix} 4 \\ -1 \end{pmatrix} = T\begin{pmatrix} 1 \\ -2 \end{pmatrix} + T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

b) Our A must be 4×2, so that Col A is a subspace $\mathbf{R}^4$ and Nul A is a subspace of $\mathbf{R}^2$. The line $y = 10x$ is spanned by $\begin{pmatrix} 1 \\ 10 \end{pmatrix}$. Therefore, if $\begin{pmatrix} a & b \end{pmatrix}$ is a row of A, then

$$0 = \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} = a + 10b.$$

Thus, $a = -10b$ and $b = b$, so a row of A is $\begin{pmatrix} -10 & 1 \end{pmatrix}$ or any scalar multiple of it. We need exactly one free variable since Nul A is a line.

$$A = \begin{pmatrix} -10 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$ is one example.

[Scratch work]