Math 1553 Exam 2, SOLUTIONS, Fall 2025, Version A

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25 AM) Kim (B, 8:00 AM) Kim (C, 9:00 AM)

Callis (D, 10:00 AM) Short (E, 9:30 AM) Shi (F, 11:00 AM)

Short (H, 12:30 PM) He (I, 2:00 PM) Stokolosa (L, 3:30 PM)

Van Why (M, 3:30 PM) Yap (N, 5:00 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, October 15.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) Suppose A is an $m \times n$ matrix. If A has a row of zeros, then the columns of A must be linearly dependent.

   ○ True

   ● False

   (b) Let V be the subspace of $\mathbf{R}^3$ consisting of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ satisfying the equation

   $$5x_1 + x_2 + x_3 = 0.$$

   Then $\left\{ \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for V.

   ● True

   ○ False

   (c) If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation and $n < m$, then T cannot be one-to-one.

   ○ True

   ● False

   (d) If A is an $m \times n$ matrix, then its null space must be a subspace of $\mathbf{R}^n$.

   ● True

   ○ False

   (e) Suppose A is a 4×2 matrix and B is a 2×8 matrix, and let T be the linear transformation given by $T(x) = ABx$. Then the domain of T is $\mathbf{R}^8$ and the codomain of T is $\mathbf{R}^4$.

   ● True

   ○ False

Problem 1 Solution.

(a) False: for example, the matrix below has a row of zeros, but its columns are linearly independent.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) True. We can see V is a plane through the origin in $\mathbf{R}^3$ from way back in section 1.1 or by observing that $V = \text{Nul}\begin{pmatrix} 5 & 1 & 1 \end{pmatrix}$, so $\dim(V) = 2$. The vectors $\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ are linearly independent, and we can verify by hand that they are both in V since $5(1) - 4 - 1 = 0$ and $0 - 1 + 1 = 0$, so they form a basis of V by the Basis Theorem.

(c) False: for example, the transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x, y) = (x, y, 0)$ is one-to-one.

(d) True: the solution set to $Ax = 0$ is a solution set of $\mathbf{R}^n$.

(e) True: AB is a 4×8 matrix, so the domain of T is $\mathbf{R}^8$ and the codomain of T is $\mathbf{R}^4$

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Find all real values of c (if there are any) so that the set below is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ 10 \end{pmatrix} \right\}$$

○ $c = 0$ only ○ $c = 1$ only ○ $c = 2$ only ○ $c = 5$ only

○ $c = -1$ only ○ $c = -2$ only ○ $c = -10$ only

○ All c except -1 ● All c except -2 ○ All c except 2

○ The set cannot be linearly independent ○ none of these

(b) (2 points) Suppose v_1, v_2, and v_3 are vectors in $\mathbf{R}^n$, and let A be the matrix whose columns are v_1, v_2, and v_3. Which **one** of the following statements guarantees that $\{v_1, v_2, v_3\}$ is linearly independent?

○ No column of A is a scalar multiple of any other column of A.

○ The zero vector is a solution to $Ax = 0$.

● For every vector b in Span$\{v_1, v_2, v_3\}$, the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ has exactly one solution.

○ A has three rows.

(c) (3 points) Let V be the set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ that satisfy $y \geq 3x$. Which of the subspace properties does V satisfy? Fill in the bubble for all that apply.

● V contains the zero vector.

● V closed under addition. In other words, if u and v are vectors in V, then $u + v$ must be in V.

○ V closed under scalar multiplication. In other words, if u is a vector in V and c is a scalar, then cu must be in V.

(d) (3 points) Let W be a 3-dimensional subspace of $\mathbf{R}^5$. Which of the following statements must be true? Fill in the bubble for all that apply.

● Every basis of W has exactly 3 vectors.

○ Every vector in $\mathbf{R}^5$ is a linear combination of vectors in W.

● If $\{a, b, c\}$ is a linearly independent set of vectors in W, then $\{a, b, c\}$ must be a basis for W.

Problem 2 Solution.

(a) We row-reduce: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & c \\ 0 & 5 & 10 \end{pmatrix} \xrightarrow{R_3 = R_3 + 5R_2} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-1} & c \\ 0 & 0 & \boxed{10+5c} \end{pmatrix}$.

Therefore, we have a pivot in every column unless $c = -2$, therefore the vectors are linearly independent for $c \neq -2$.

(b) Statement (i) does not guarantee linear independence, for example the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Statement (ii) does not guarantee linear independence, since the zero vector is **always** a solution to $Ax = 0$, no matter what.

Statement (iii) guarantees linear independence. The zero vector is in the span of v_1, v_2, and v_3, so the statement guarantees that $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has only one solution (namely the trivial solution), therefore the vectors are linearly independent.

Statement (iv) does not really do anything. For example, the 3×3 zero matrix satisfies the statement, but its columns are obviously linearly dependent.

(c) It is probably easiest to graph V. It is the triangular region of all points that are on, or above, the line $y = 3x$.

- Yes, V contains the zero vector since $0 \geq 3(0)$.
- Yes, V is closed under addition. Adding any two vectors in V keeps them in V. We could argue algebraically too: if $y_1 \geq 3x_1$ and $y_2 \geq 3x_2$, then for the vector $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$, we get $y_1 + y_2 \geq 3x_1 + 3x_2 = 3(x_1 + x_2)$, so its second coordinate is greater than or equal to 3 times its first coordinate.
- No, V is not closed under scalar multiplication. What happens is that multiplying by a negative number flips the inequality sign. For example, $u = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ is in V because $5 > 3(1)$, but $-u = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ is not in V since $-5 < 3(-1)$.

(d) Statement (i) is true: $\dim(W)$ is the number of vectors in any basis for W.

Statement (ii) is not true. Every linear combination of vectors in W is another vector in W since it is a subspace, and many vectors in $\mathbf{R}^5$ are not in the 3-dimensional subspace W.

Statement (iii) is true by the Basis Theorem: any 3 linearly independent vectors in a 3-dimensional subspace automatically form a basis for the subspace.

3. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Find the **one** matrix A below that satisfies

$$\text{Col}(A) = \text{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\} \quad \text{and} \quad \text{Nul}(A) = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}.$$

○ $A = \begin{pmatrix}1 & -2\\4 & -8\end{pmatrix}$ ○ $A = \begin{pmatrix}1 & 1\\4 & 2\end{pmatrix}$ ● $A = \begin{pmatrix}-2 & 1\\-8 & 4\end{pmatrix}$

○ $A = \begin{pmatrix}1 & 1/2\\4 & 2\end{pmatrix}$ ○ $A = \begin{pmatrix}2 & 1\\8 & 4\end{pmatrix}$ ○ $A = \begin{pmatrix}1 & -1/2\\8 & -4\end{pmatrix}$

○ none of these

(b) (3 points) Which of the following transformations are linear? Fill in the bubble for all that apply.

● $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (3x_1 - x_2,\ x_1)$

○ $T : \mathbf{R}^2 \to \mathbf{R}$ given by $T(x, y) = x + 1$.

● $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\ln(7) & 0\\ \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$.

(c) (2 points) Suppose A is a 45×50 matrix and there are 20 free variables in the solution set to the homogeneous equation $Ax = 0$. Which **one** of the following statements is true about the column space of A?

○ Col(A) is a 20-dimensional subspace of $\mathbf{R}^{50}$.

○ Col(A) is a 25-dimensional subspace of $\mathbf{R}^{50}$.

○ Col(A) is a 30-dimensional subspace of $\mathbf{R}^{50}$.

○ Col(A) is a 25-dimensional subspace of $\mathbf{R}^{45}$.

● Col(A) is a 30-dimensional subspace of $\mathbf{R}^{45}$.

(d) (3 points) Which of the following linear transformations are **onto**? Fill in the bubble for all that apply.

○ The transformation $T(x) = Ax$, where $A = \begin{pmatrix}1 & 3 & -4 & 0\\1 & 1 & 0 & 1\\0 & 2 & -4 & -1\end{pmatrix}$.

○ The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1 - 2x_2,\ 3x_1 - 6x_2)$

● The transformation $T(x) = Ax$, where A is a 4×5 matrix whose null space is a line.

Problem 3 Solution.

(a) Each column of A must be a scalar multiple of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Only two options satisfy $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, namely

$$\begin{pmatrix} -2 & 1 \\ -8 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1/2 \\ 8 & -4 \end{pmatrix}.$$

However, the second matrix above has the wrong column span, so the only matrix that satisfies both conditions is $A = \begin{pmatrix} -2 & 1 \\ -8 & 4 \end{pmatrix}$.

(b) (i) is linear, since T is the matrix transformation $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$.
(ii) is not linear, which we can see immediately since $T(0,0) = 1 \neq 0$.
(iii) is linear, since it is a matrix transformation. The numbers $\ln(7)$ and $\sqrt{2}$ are just real scalars.

(c) This is a quintessential application of the Rank Theorem. We are told that Nul(A) is 20-dimensional, and A has 50 columns, so:

$$\dim(\mathrm{Col(A)}) + \dim(\mathrm{Nul(A)}) = 50, \qquad \dim(\mathrm{Col(A)}) + 20 = 50, \qquad \dim(\mathrm{Col(A)}) = 30.$$

Also, since A has 45 rows we know $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^{45}$.

(d) The T in (i) is not onto because A row-reduces to $\begin{pmatrix} 1 & 3 & -4 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which fails to have a pivot in its third row.

The T in (ii) is not onto because its standard matrix is $A = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$, which row-reduces to $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$, so its range is just a line in $\mathbf{R}^2$.

The T in (iii) is onto: if the null space of a 4×5 matrix is a line, then A has exactly one column without a pivot, therefore it has 4 pivots. This means A has a pivot in every row, so T is onto.

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ across the y-axis, then reflects across the line $y = x$. Solve $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

○ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ● $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ○ there is no solution

(b) (2 points) Suppose a linear transformation T satisfies

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Compute $T(v)$ for the vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

● $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$ ○ $\begin{pmatrix} -9 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} 3 & -4 \\ 4 & 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

○ $\begin{pmatrix} 11 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & -10 \\ 2 & 2 \end{pmatrix}$ ○ none of these

(c) (4 points) Let $T : \mathbf{R}^s \to \mathbf{R}^t$ be a linear transformation with standard matrix A. Which of the following statements guarantee that T is one-to-one? Fill in the bubble for all that apply.

● For every y in $\mathbf{R}^t$, there is at most one x in $\mathbf{R}^s$ so that $T(x) = y$.

○ For every x in $\mathbf{R}^s$, there is at most one y in $\mathbf{R}^t$ so that $T(x) = y$.

● The range of T is an s-dimensional subspace of $\mathbf{R}^t$.

● The matrix equation $Ax = 0$ has only the trivial solution.

(d) (2 points) Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & h \end{pmatrix}$. Find all real values of h (if there are any) so that the linear transformation $T(x) = Ax$ is onto.

○ $h = 0$ only ○ $h = 1$ only ○ All h except 1 ○ All h except 5

● The transformation T cannot be onto, no matter what h is.

Problem 4 Solution.

(a) We need to solve $ABv = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reflection across $y = x$ and the matrix $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is reflection across the y-axis.

The reason why B is on the right is because it is the **first** operation done: if v is in $\mathbf{R}^2$ then we first take Bv, then take $A(Bv)$.

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we solve:

$$ABv = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right).$$

This gives $x = -1$ and $y = 0$, so the answer is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

(b) $v = x - 2y$ where $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, so by linearity:

$$T(v) = T(x - 2y) = T(x) - T(2y) = T(x) - 2T(y) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}.$$

(c) Statement (i) is nearly verbatim the definition of one-to-one.

Statement (ii) does not guarantee T is one-to-one. In fact, if T is **any** transformation and x is in its domain, then there can never be more than one value assigned to $T(x)$.

Statement (iii) guarantees T is one-to-one, since this means $\dim(\text{Col } A) = s$ and therefore the $t \times s$ matrix A has a pivot in every column.

Statement (iv) guarantees T is one-to-one, since it means A has a pivot in every column.

(d) The matrix is 3×2, so immediately we conclude it has at most 2 pivots, so the range cannot be larger than a plane in $\mathbf{R}^3$. Therefore, T cannot be onto, no matter what h is.

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, consider the matrix A and its reduced row echelon form given below.

$$A = \begin{pmatrix} 2 & 6 & -8 & 3 \\ 1 & 3 & -4 & 1 \\ -1 & -3 & 4 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (2 pts) Write a basis for Col(A). You do not need to show your work on this part.

Solution: We can choose the pivot columns, which are the first and fourth:

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \right\}.$$

Alternatively, we could have used the fourth column along with any one of the first three columns, since the first three are all nonzero scalar multiples of each other. Many answers are possible.

(b) (3 points) Write a new basis for Col(A), so that no vector in your new basis is a scalar multiple of any of the vectors you wrote in part (a). Clearly show how you obtain this basis.

Solution: We can take linear combinations of the answer from part (a). We just need to be sure our answers are linearly independent and that they are not scalar multiples of anything from (a). If $u = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, then our new basis can be $\{u + v,\ u - v\}$ which is

$$\left\{ \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \right\}.$$

Many other answers are possible. For example, we can take $\{2u + v,\ u + 2v\}$ which is

$$\left\{ \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \\ -1 \end{pmatrix}, \right\}.$$

(c) (4 points) Find a basis for Nul(A).

Solution: We are given the RREF of A, so we know the RREF of $(A|0)$. This gives us $x_1 + 3x_2 - 4x_3 = 0$ and $x_4 = 0$, where x_2 and x_3 are free. Therefore, $x_1 = -3x_2 + 4x_3$ and we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 + 4x_3 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is therefore $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(d) (1 pt) Find one x so that $Ax = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Write your answer here: $x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$.

You do not need to show your work on this part, and there is no partial credit.

Solution: Note that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is the fourth column of A, so $A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, and we get the solution $x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ without needing to do any work.

Alternatively, we could solve the whole thing and we would find that ANY vector of the form

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is a solution. However, for credit, the student cannot just write the parametric vector form, they must write **one** specific vector x as required by the problem.

6. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit.

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation $T(x_1, x_2, x_3) = (x_1 - x_2,\ 4x_1 + 3x_3)$ and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors in $\mathbf{R}^2$ by $45°$ counterclockwise.

(a) Find the standard matrix A for T and write it in the space below. Show your work.

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 4 & 0 & 3 \end{pmatrix}.$$

(b) Write the standard matrix B for U. Evaluate any trigonometric functions you write. Do not leave your answer in terms of sine and cosine.

$$B = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

There are other ways to write B, for example

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

(c) Which composition makes sense: $T \circ U$ or $U \circ T$? Fill in the correct bubble below. You do not need to show your work on this part.

○ $T \circ U$ ● $U \circ T$

(d) Compute the standard matrix C for the composition you selected in (c). Put your answer in the space provided below.
Solution:

$$BA = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 4 & 0 & 3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -3 & -1 & -3 \\ 5 & -1 & 3 \end{pmatrix}.$$

Other equivalent correct answers are

$$BA = \begin{pmatrix} \frac{-3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} \quad \text{or} \quad BA = \begin{pmatrix} \frac{-3\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & \frac{-3\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \end{pmatrix}.$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit. Parts (a) through (c) are unrelated.

   (a) (3 points) Let V be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form

$$\begin{pmatrix} -4a+3b \\ b \\ a+b \\ b \end{pmatrix},$$

   where a and b are real. Find a basis for V. Enter it in the space provided below.
   Solution: We find a basis as follows.

$$\begin{pmatrix} -4a+3b \\ b \\ a+b \\ b \end{pmatrix} = \begin{pmatrix} -4a \\ 0 \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} 3b \\ b \\ b \\ b \end{pmatrix} = a\begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \qquad \text{Basis}: \left\{ \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

   (b) (3 points) Let T be the linear transformation $T(x) = Ax$, where A is the matrix $A = \begin{pmatrix} 4 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Find $T\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Enter your answer in the space below.
   Solution: We just compute:

$$T\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4(1)-1(2)+3(-1) \\ 0(1)+1(2)+2(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

   If the student wrote a 2×3 (or similar) matrix as their answer, or wrote a vector in $\mathbf{R}^3$, they likely received 0 points. There is simply no reason for writing anything other than a vector in $\mathbf{R}^2$ as the answer.

   (c) (4 points) Suppose A is a 2×2 matrix whose null space is Span $\left\{\begin{pmatrix} -4 \\ 1 \end{pmatrix}\right\}$.
   Find a 2×2 matrix B that has at least one nonzero entry and satisfies:

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Enter your answer in the space below.
   Solution: Call v_1 and v_2 the columns of B. By definition of matrix multiplication,

$$AB = \begin{pmatrix} Av_1 & Av_2 \end{pmatrix}.$$

   Therefore, in order for AB to the zero matrix, we just need to choose v_1 and v_2 to be vectors in Nul(A). In other words, we just need to choose v_1 and v_2 to be scalar multiples of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$, and at least one of them must not be the zero vector. There are infinitely correct answers. We will list some below.

$$B = \begin{pmatrix} -4 & -4 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -8 & 4 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 16 \\ 0 & -4 \end{pmatrix}, \text{ etc.}$$

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