MATH 1553, JANKOWSKI (A1-A6)
MIDTERM 1, FALL 2018

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1. [10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer.

a) **T** **F** The matrix $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$ is in reduced row echelon form.

b) **T** **F** If the bottom row of an augmented matrix in RREF is $\left(\begin{array}{ccc|c} 0 & 1 & 2 & 3 \end{array}\right)$, then the corresponding system of equations must be consistent.

c) **T** **F** The vector equation $x\begin{pmatrix} 1 \\ -1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has infinitely many solutions.

d) **T** **F** If A is a 3×4 matrix and b is a vector so that the set of solutions to $Ax = b$ is a line through the origin, then $b = 0$.

e) **T** **F** If A is a 3×3 matrix and $Ax = 0$ has infinitely many solutions, then $Ax = b$ must be inconsistent for some b in $\mathbf{R}^3$.

Solution.

a) False. The entry above the pivot in the final column is nonzero.

b) True. With the bottom row in view, we see that the rightmost pivot in the system is to the left of the augment, so the right column cannot be a pivot. In fact, the system has infinitely many solutions! I took this almost directly from a Webwork T/F.

c) False. One step row-reduction gives the unique solution $x = 1$, $y = 0$.

d) True. If the solution set contains the origin then $x = 0$ is a solution so $b = A(0) = 0$. Very similar to a 3.3-3.4 supplemental problem.

e) True. If $Ax = 0$ has infinitely many solutions then A has at most two pivots. Since A has three rows, this means A cannot have a pivot in every row, so $Ax = b$ must be inconsistent for some b in $\mathbf{R}^3$.

Extra space for scratch work on problem 1

Problem 2.

[10 points]

You do not need to show your work in parts (a) through (c).

a) [2 points] Suppose we have a consistent system of four linear equations in three variables, and the corresponding augmented matrix has two pivots. The set of solutions to the system is a:

(circle one answer) **point** $\boxed{\textbf{line}}$ **plane** **3-plane**

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\boxed{\mathbf{R}^3}$ $\mathbf{R}^4$.

b) [2 pts] Which of the following conditions guarantee that a system of 3 linear equations in 4 variables has *infinitely many solutions*? (circle all that apply)
(1) The reduced row echelon form of the augmented matrix corresponding to the system of equations has a row of zeros. (system can still be inconsistent)
$\boxed{(2)}$ The rightmost column of the augmented matrix is not a pivot column. (system consistent, and at most three pivots so one or more free variables)

c) [2 pts] Write a vector v in $\mathbf{R}^3$ which is NOT a linear combination of $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}0\\-5\\0\end{pmatrix}$.

<u>Answer</u>: Each of the vectors $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}0\\-5\\0\end{pmatrix}$ has first entry equal to third entry, so any vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ satisfying $x_1 \neq x_3$ will not be a linear combination of those two vectors. For example, $v = \begin{pmatrix}1\\0\\2\end{pmatrix}$.

d) [4 pts] Write a matrix A, with the property that $Ax = b$ is consistent if and only if b is in the span of $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\3\end{pmatrix}$. Briefly justify your answer.

<u>Answer</u>: $Ax = b$ is consistent if and only if b is in the span of the columns of A, so just let

$$A = \begin{pmatrix}0 & -1\\1 & 0\\-1 & 3\end{pmatrix}.$$

Extra space for work on problem 2

Problem 3. [9 points]

Michael Scarn is convinced that the key to his mystery is contained in the system of equations

$$x - ky = 3$$
$$-2x - 4y = h,$$

where h and k are real numbers.

a) For what values of h and k (if any) does the system have a unique solution?

b) For what values of h and k (if any) does the system have infinitely many solutions?

Solution.

The system row-reduces to

$$\left(\begin{array}{cc|c} 1 & -k & 3 \\ -2 & -4 & h \end{array}\right) \xrightarrow{R_2=R_2+2R_1} \left(\begin{array}{cc|c} 1 & -k & 3 \\ 0 & -4-2k & h+6 \end{array}\right).$$

a) The system has a unique solution when it is consistent and has no free variables. This means $-4-2k \neq 0$ (so $k \neq -2$) and h can be any real number. $\boxed{k \neq -2 \text{ and } h \text{ is any real number}}$.

b) The system has infinitely many solutions when there is at least one free variable and the right column is not a pivot column. This means $-4-2k = 0$ and $h+6=0$. $\boxed{k=-2 \text{ and } h=-6}$.

Extra space for work on problem 3

Problem 4. [10 points]

Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$x_1 - 3x_3 + x_4 = 1$$
$$3x_1 + x_2 - 9x_3 + 5x_4 = 1$$
$$-2x_1 + 2x_2 + 6x_3 + 2x_4 = -6.$$

a) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

b) Write the set of solutions to the system from part (a) in parametric vector form.

c) Write *one* vector which is not the zero vector and solves the corresponding homogeneous system of linear equations:

$$x_1 - 3x_3 + x_4 = 0$$
$$3x_1 + x_2 - 9x_3 + 5x_4 = 0$$
$$-2x_1 + 2x_2 + 6x_3 + 2x_4 = 0.$$

Solution.

a)

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 3 & 1 & -9 & 5 & 1 \\ -2 & 2 & 6 & 2 & -6 \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2-3R_1} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 2 & 0 & 4 & -4 \end{array}\right) \xrightarrow{R_3=R_3-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

b) From (a) we see the solutions have the form

$$x_1 = 1 + 3x_3 - x_4, \qquad x_2 = -2 - 2x_4, \qquad x_3 = x_3 (\text{free}), \qquad x_4 = x_4\ (\text{free}).$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 3x_3 - x_4 \\ -2 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ -2x_4 \\ 0 \\ x_4 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}}.$$

c) The vectors associated to x_3 and x_4 in (b) are non-trivial homogenous solutions. So, for example, $\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ are correct, as is any linear combination of the two except the zero vector. The answers $x_3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $x_4 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ are not totally correct. For example, $x_3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ doesn't just describe one vector (since x_3 is a free variable), and if $x_3 = 0$ then it is the zero vector.

Extra space for work on problem 4

Problem 5.

[11 points]

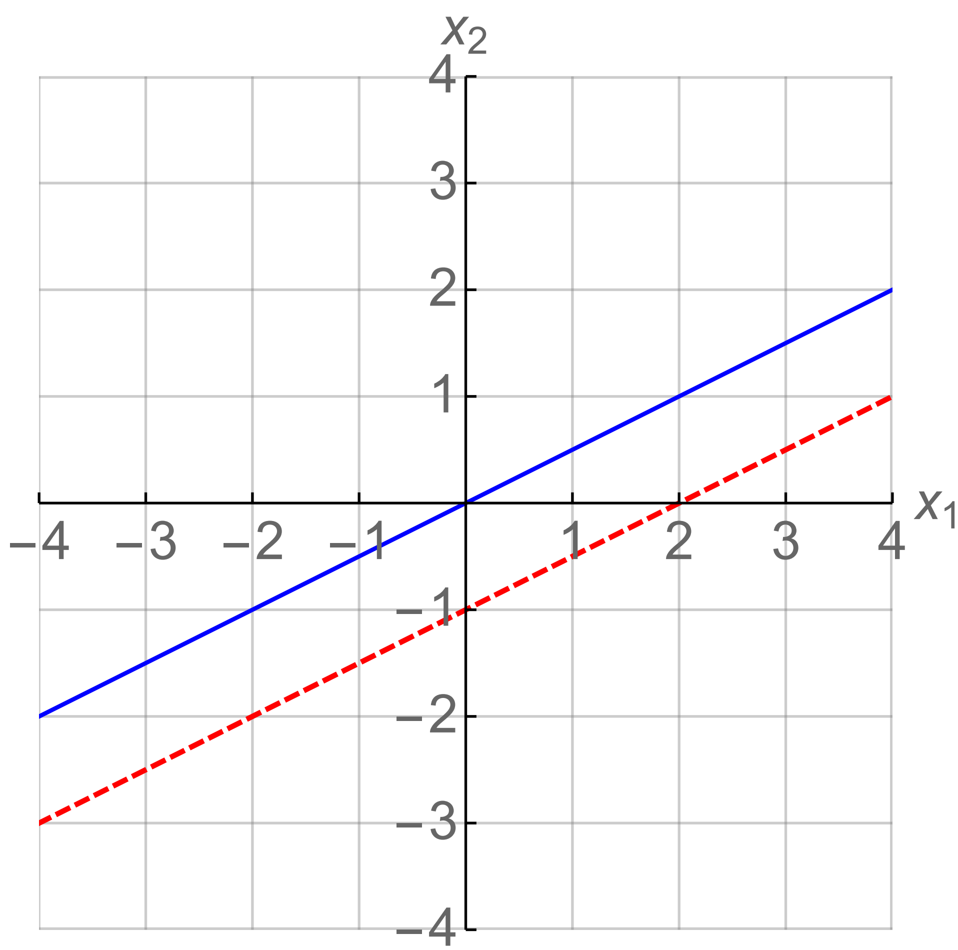
Throughout this problem, let $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$.

a) Draw the solution set for $Ax = 0$ and draw the solution set for $Ax = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ on the same graph below. Clearly label each solution set.

$$\left(\begin{array}{cc|c} 2 & -4 & 0 \\ -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} 2 & -4 & 4 \\ -1 & 2 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

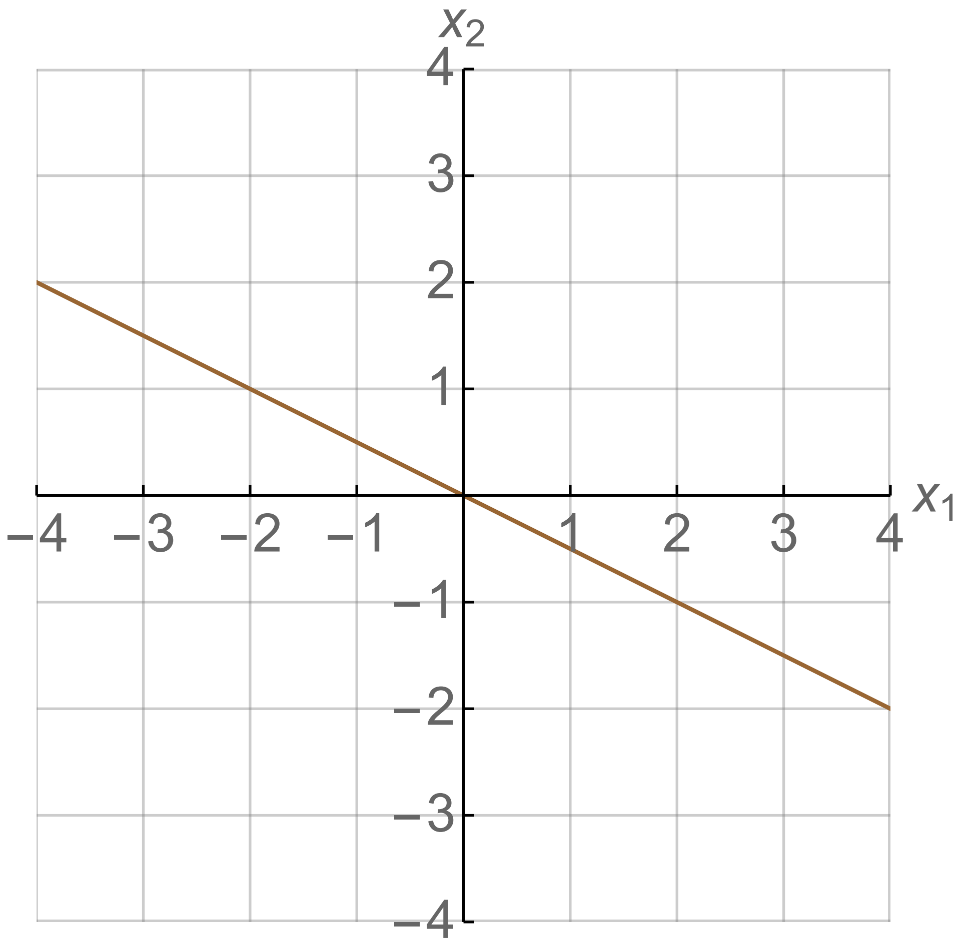
$Ax = 0$ has solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, giving Span $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$. (blue)

$Ax = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ has solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. (graphed dashed red)



b) Draw the span of the columns of A on the graph below.

Each column of A is a nonzero scalar multiple of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, so the column span is just the line through the origin and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.



c) Is there a vector b so that $Ax = b$ has a unique solution? Justify your answer.

No. By section 3.4 we know that the solution set to any consistent system $Ax = b$ will have a solution is a translation of the solution set to $Ax = 0$, which is a line. Therefore, if $Ax = b$ is consistent then it must have infinitely many solutions. Similarly one could argue using pivots, since if $(A|b)$ is consistent then it will only have one pivot on the left side and thus a free variable, yielding infinitely many solutions.

Extra space for work on problem 5