Quiz 8, Discrete Math (15 points), Fall 2016

The quiz is 20 minutes. Show your work and justify your answers where appropriate. If you write the correct answer without sufficient work or justification, you will receive little or no credit.

1. (1 point each) Fill in the blank. No justification is needed and no partial credit will be given. As in class, complexity is measured by the number of comparisons required (and is written in terms of n).

   (a) The Linear Search Algorithm for finding a number x in a list of n numbers $a_1, a_2, \ldots, a_n$, has complexity $\mathcal{O}(\_\_\_\_\_\_\_\_\_\_)$. Answer: $\mathcal{O}(n)$.

   (b) The Binary Search Algorithm for finding a number x in a list of n numbers $a_1 \leq a_2 \leq \cdots \leq a_n$, has complexity $\mathcal{O}(\_\_\_\_\_\_\_\_\_\_)$. Answer: $\mathcal{O}(\log_2(n))$. (it's fine if they write "$\log n$")

   (c) The Bubble Sort Algorithm for sorting a list of n numbers $a_1, a_2, \ldots, a_n$ into ascending order, has complexity $\mathcal{O}(\_\_\_\_\_\_\_\_\_\_)$. Answer: $\mathcal{O}(n^2)$

   (d) The Merge Sort Algorithm for sorting a list of n numbers $a_1, a_2, \ldots, a_n$ into ascending order, has complexity $\mathcal{O}(\_\_\_\_\_\_\_\_\_\_)$. Answer: $\mathcal{O}(n \log_2(n))$. (it's fine if they write "$\log n$" rather than $\log_2 n$)

2. (4 points) Use the Binary Search Algorithm to search for $x = 8$ in the list $3, 8, 10, 14, 18, 20, 21$. You should make sure your steps are clear. However, you do not need to write paragraphs explaining your steps.

   Solution: There are $n = 7$ numbers. We take $\lfloor \frac{7}{2} \rfloor = 3$, $a_3 = 10$. Since $8 \leq 10$, we reduce the list to $3, 8, 10$.

   Now $n = 3$. We take $\lfloor \frac{3}{2} \rfloor = 1$, $a_1 = 3$. Since $8 \nleq 3$, we reduce the list to $8, 10$.

   Now $n = 2$. We take $\lfloor \frac{2}{2} \rfloor = 1$, $a_1 = 8$. Since $8 \leq 8$, we reduce the list 8 only.

   Now $n = 1$, so we test equality: $8 = 8$.

3. (4 pts) How many integers between 1 and 112 (including 1 and 112) are relatively prime to 112? (hint: $112 = 2^4 \cdot 7$)

Solution: An integer is relatively prime to 112 if and only if it is neither divisible by 2 nor divisible by 7. $A = \{n \in \mathbb{N} \ : \ 1 \le n \le 112, \ 2|n\}$ and $B = \{n \in \mathbb{N} \ : \ 1 \le n \le 112, \ 7|n\}$. We want $|(A \cup B)^c|$.

$$\begin{aligned}|(A \cup B)^c| &= |U| - |A \cup B| = |U| - (|A| + |B| - |A \cap B|) \\ &= 112 - \left(\lfloor \frac{112}{2} \rfloor + \lfloor \frac{112}{7} \rfloor - \lfloor \frac{112}{14} \rfloor\right) \\ &= 112 - (56 + 16 - 8) = 112 - 64 = \boxed{48}.\end{aligned}$$

4. (3 pts) How many integers between 1 and 150 (including 1 and 150) are divisible by 3 or by 7, but not both?

Solution: Here we use $A = \{n \in \mathbb{N} \ : \ 1 \le n \le 150, \ 3|n\}$ and $B = \{n \in \mathbb{N} \ : \ 1 \le n \le 150, \ 7|n\}$.
We can find $|A \cup B|$ and then subtract $|A \cap B|$, or we can simply use the symmetric difference formula from the beginning (exactly the same thing algebraically).

Option 1: $|A \cup B| = |A| + |B| - |A \cap B| = \lfloor \frac{150}{3} \rfloor + \lfloor \frac{150}{7} \rfloor - \lfloor \frac{150}{21} \rfloor = 50 + 21 - 7 = 64.$
Therefore, our final answer is $|A \cup B| - |A \cap B| = 64 - 7 = \boxed{57}$.

Option 2: Using the symmetric difference formula,

$$|A \oplus B| = |A| + |B| - 2|A \cap B| = \lfloor \frac{150}{3} \rfloor + \lfloor \frac{150}{7} \rfloor - 2\lfloor \frac{150}{21} \rfloor = 50 + 21 - 14 = \boxed{57}.$$