Quiz 1, Discrete Math (15 points), Fall 2016

Show your work and justify your answers where appropriate. If you write the correct answer without sufficient work or justification, you will receive little or no credit.

1. (1 point each) Clearly circle your answer (no justification needed here, and no partial credit given).

   (a) If an implication is true, then its converse must also be true. YES $\boxed{\text{NO}}$

   (b) If 8 is a prime number, then $5^2 = 16$. $\boxed{\text{TRUE}}$ FALSE

   (c) $2 + 2 = 5$ or $\sqrt{7}$ is rational. TRUE $\boxed{\text{FALSE}}$

2. (3 points) Write the negation of the following statement using quantifiers (do not use language like "it is not the case that..." or "there is no..."):

   "For every real number x, there is a real number y that satisfies $y > x$ and $y - x < 1$."

   A variety of similar correct answers are possible. Some examples:
   "There exists a real number x such that for every real number y, we have $y \leq x$ or $y - x \geq 1$."

   Or:

   "For some real number x, every real number y satisfies $y \leq x$ or $y - x \geq 1$."

   Or:

   "There exists a real number x such that every real number y satisfies $y \leq x$ or $y - x \geq 1$."

   Or in different order:

   "There is a real number x with the property that $y \leq x$ or $y - x \geq 1$ for every real number y."

3. (3 points) Is it possible for "p or (not q)" and "(not p) and q" to *both* be false? Justify your answer.

   No. If "p or (not q)" is false, then p is false and q is true, so "(not p) and q" is true.

4. (6 points) Prove that if x is a nonzero rational number and y is an irrational number, then xy is irrational.

Our proof is by contradiction. Suppose x is a nonzero rational number and y is an irrational number, but that xy is rational. Since x and xy are rational, we have $x = \frac{k}{\ell}$ and $xy = \frac{m}{n}$ for some integers k, ℓ, m, n, where $\ell \neq 0$ and $n \neq 0$. Also, $k \neq 0$ since $x \neq 0$. We find

$$x \cdot y = xy \implies \frac{k}{\ell} y = \frac{m}{n} \implies y = \frac{\ell m}{kn},$$

where $kn \neq 0$ since $k \neq 0$ and $n \neq 0$. Since ℓm and kn are integers, it follows that y is rational, which contradicts the fact that y is irrational.

Some students assumed for contradiction that y was rational, and proved that xy is rational in that case. This does not prove #4, as it only proves that the product of rational numbers is rational. Proofs by contradiction assume $\neg A$, then derive an absurdity that shows $\neg A$ cannot be true. Here, the assertion A is an if-then statement, so $\neg A$ would mean that the hypothesis (x nonzero rational and y irrational) is true, but that the conclusion (xy irrational) is false.