Math 1553 Worksheet §§2.4, 2.5

Solutions

1. Find the set of solutions to $x_1 - 3x_2 + 5x_3 = 0$. Next, find the set of solutions to $x_1 - 3x_2 + 5x_3 = 3$. In each case, write your solution in parametric vector form. How do the solution sets compare geometrically?

Solution.

The homogeneous system $x_1 - 3x_2 + 5x_3 = 0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$, which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \text{ (free)} \qquad x_3 = x_3 \text{ (free)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

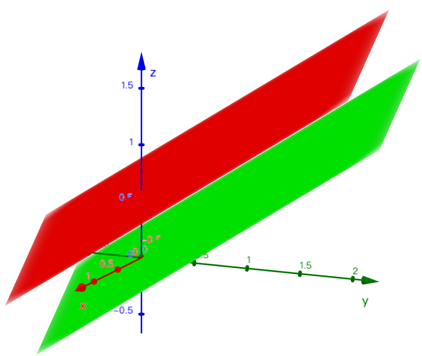
The solution set for $x_1 - 3x_2 + 5x_3 = 0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The nonhomogeneous system $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set (red) is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane (green) spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.



Here is the link to a 3D picture you can play with https://www.geogebra.org/calculator/j57ttsnb

2. If the statement is always true, circle TRUE. Otherwise, circle FALSE. Justify your answer.

a) Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then v_1, v_2, v_3 are linearly dependent? If true, write a linear dependence relation for the vectors.
TRUE **FALSE**

b) If $Ax = b$ is consistent, then $Ax = 5b$ is consistent. **TRUE** **FALSE**

c) In the following, A is an $m \times n$ matrix.

(1) **TRUE** **FALSE** If A has linearly dependent columns, then $m < n$.

(2) **TRUE** **FALSE** If A has linearly independent columns, then $Ax = b$ always have at least one solution for any b in $\mathbf{R}^m$.

(3) **TRUE** **FALSE** If b is a vector in $\mathbf{R}^m$ and $Ax = b$ has a exactly one solution, then $m \geq n$.

Solution.

a) **TRUE**. By definition of matrix multiplication, $-3v_1+2v_2+7v_3 = 0$, so $\{v_1, v_2, v_3\}$ is linearly dependent and the equation gives a linear dependence relation.

b) **TRUE**. Let v be a solution to $Ax = b$, so $Av = b$. Then $A(5v) = 5Av = 5b$.

c) (1) **FALSE** For example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(Note that even though this part was false, there is a very similar-sounding statement that is true: if $m < n$, A must have linearly dependent columns.)

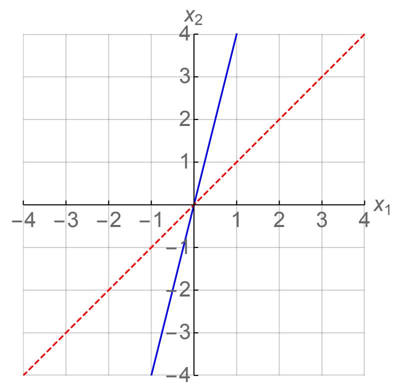
(2) **FALSE** For example $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. There is no solution for $Ax = b$.

(Note, however: if A has linearly independent columns, then the system $Ax = 0$ has no free variables, so $Ax = b$ is either inconsistent or has a unique solution.)

(3) **True** If $Ax = b$ has a unique solution, then since it is a translation of the solution set to $Ax = 0$, this means that $Ax = 0$ has only the trivial solution (no free variables). Thus, A has a pivot in every column, which is impossible if $m < n$ (i.e. impossible if A has more columns than rows), so $m \geq n$.

3. Let $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$. Draw the span of the columns of A, and draw the set of solutions to $Ax = 0$. Clearly label each.

Solution.



The blue line is the span of columns of A: Span $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$. If you draw the two column vectors, you will see they both fall on the line $x_2 = 4x_1$.

The red dashed line is the span of solutions of $Ax = 0$: Span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. To see this is the case, you can row reduce the augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$. That implies the solution set is the line $x_2 = x_1$.

4. Write an augmented matrix corresponding to a system of two linear equations in the three variables x_1, x_2, x_3, so that the solution set is the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$.

Solution.

We are asked to come up with a system whose solution set is the prescribed span, rather than being handed a system and discovering its solution set.

Since the span of any vector includes the origin, the zero vector is a solution, so the system is homogeneous.

Note that the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ is all vectors of the form $t \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ where t is real.

It consists of all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ so that $x_1 = -4x_2,\ \ x_2 = x_2,\ \ x_3 = 0$.

The equation $x_1 = -4x_2$ gives $x_1 + 4x_2 = 0$, so one line in the matrix can be $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \end{array}\right)$.

The equation $x_3 = 0$ translates to $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \end{array}\right)$. Note that this leaves x_2 free, as desired.

This gives us the augmented matrix

$$\boxed{\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)}.$$

(Multiple examples are possible. For example do an arbitrary row operation on the above matrix, that will also work.)