Math 1553 Worksheet §2.1, §2.2, §2.3

Solutions

1. **a)** Write a set of three vectors whose span is a point in $\mathbf{R}^3$.

b) Write a set of three different vectors whose span is a line in $\mathbf{R}^3$.

c) Write a set of three different vectors whose span is a plane in $\mathbf{R}^3$.

d) In each of the above questions, if you use the three vectors form a matrix A, how many pivots does A have?

Solution.

a) The span of any three vectors v_1, v_2, v_3 in $\mathbf{R}^3$ must contain the origin, since

$$0v_1 + 0v_2 + 0v_3 \text{ is automatically the zero vector.}$$

There is only one possibility for this answer: we must choose $v_1 = v_2 = v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. If our list had contained a nonzero vector, then the span would include that nonzero vector and all scalar multiples of it (including the zero vector).

b) Just choose any vector that span your favorite line, then pick the other vectors to be within that line. For example, choose $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, which span the x-axis within $\mathbf{R}^3$.

c) Similar to above. Just choose any two vectors that span your favorite plane, then pick your third vector to be within that plane. For example, choose $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. The span of these three vectors is the xy-plane in $\mathbf{R}^3$.

d) For a) the matrix A has no pivot. For b) the matrix has one pivot. For c) the matrix has two pivots.

2. Consider the system of linear equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= -2 \\ -x - y &= 4. \end{aligned}$$

Question: Does this system have a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a vector equation.

c) Formulate this question into a matrix equation $Av = b$.

d) What does this mean in terms of spans?

e) Answer the question using the interactive demo.

f) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right)$$

b) What are the solutions to the following vector equation?

$$x\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$$

c) As matrix equation:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}, v = \begin{pmatrix} x \\ y \end{pmatrix}, b = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$$

d) Is $\begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$ in Span $\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$?

e) The picture in the interactive demo shows that b is not in the span of the columns of A, so the system of linear equations is inconsistent.

f) From part **e** we already know the system is inconsistent, so here we confirm it using row reduction. Row reducing the matrix in part **a** yields

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),$$

so there are no solutions to the system of linear equations.

3. Catherine Halsey has challenged you to find a hidden treasure, located at some point (a, b, c). She has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding Catherine's message, you have discovered that the first and second coordinates of the treasure's location are (in order) −4 and 3.

a) What is the treasure's full location?

b) Give instructions for how to find the treasure by only moving in the directions given by v_1, v_2, and v_3.

Solution.

a) We translate this problem into linear algebra. Let c be the final entry of the treasure's location. Since Catherine has assured us that we can find the treasure using the three vectors we have been given, our problem is to find c so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is a linear combination of v_1, v_2, and v_3 (in other words, find c so that the treasure's location is in in $\text{Span}\{v_1, v_2, v_3\}$). We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right).$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c - 5 = 0$, or $c = 5$.

The location of the treasure is $(-4, 3, 5)$.

b) Getting to the point $(-4, 3, 5)$ using the vectors v_1, v_2, and v_3 is equivalent to finding scalars x_1, x_2, and x_3 so that

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

We can rewrite this as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= -4 \\ -x_1 - 4x_2 + x_3 &= 3 \\ -2x_1 - 7x_2 \phantom{{}+3x_3} &= 5. \end{aligned}$$

We put the matrix from part (a) into RREF.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1-5R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Note x_3 is the only free variable, so:

$$x_1 = 1 - 7x_3, \qquad x_2 = -1 + 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. If we choose the path corresponding to $x_3 = 0$, then

$x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of v_1 and -1 unit in the direction of v_2. In equations:

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

4. True or false. If the statement is *always* true, answer True. Otherwise, answer False. In parts (a) and (b), A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$.

a) If b is in the span of the columns of A, the matrix equation $Ax = b$ is consistent.

b) A does not have a pivot in every column if $Ax = b$ is inconsistent.

c) If A is a 4×3 matrix, then the equation $Ax = b$ is inconsistent for some b in $\mathbb{R}^4$.

Solution.

a) True. Let the columns of A be $v_1, \cdots, v_n$. Since b in Span$\{v_1, \cdots, v_n\}$, this means b can be written as a linear combinations of these column vectors, so

$$x_1 v_1 + \cdots + x_n v_n = b$$

for some scalars $x_1, \ldots, x_n$. Therefore, $Ax = b$ where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

b) False, for instance consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is an inconsistent system even though A has a pivot in each column.

c) True. For example if we take the augmented matrix $\left(A \mid b \right)$ as

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

In fact, any 4×3 matrix A will have at most 3 pivots in A, so we can choose b to make sure that it has a pivot as well in the augmented matrix $\left(A \mid b \right)$.