Math 1553 Worksheet: Fundamentals and §1.1

Solutions

1. In this problem, we will discuss when you can solve a linear system and when you cannot, by using some examples. Moreover, we explore whether this is related to the number of equations and variables.

(1) If a linear system has 3 equations and 2 unknown variables, is it possible to find a solution? If your answer is yes, give an example. If your answer is maybe, give an example of a consistent system and an example of an inconsistent system.

(2) If a linear system has 2 equations and 3 unknown variables, is it possible to find a solution? If your answer is yes, give an example. If your answer is maybe, give an example of a consistent system and an example of an inconsistent system.

(3) If a linear system has 2 equations and 2 unknown variables, must it have a solution? Please explain.

Solution.

(1) Maybe.

'Yes': when one equation is a multiple of another. For example, the linear system

$$\begin{aligned} 2x + 4y &= 0 \\ x + 2y &= 0 \\ y &= 1 \end{aligned}$$

has the solution $x = -2, y = 1$

'No': when these equations form a triangle in $\mathbf{R}^2$(In other words, they are contradicting with each other). For example, the linear system

$$\begin{aligned} x &= 0 \\ x + y &= 0 \\ y &= 1 \end{aligned}$$

has the solution $x = -2, y = 1$

(2) Maybe.

'Yes': For example, the linear system

$$\begin{aligned} x + y &= 0 \\ x + y + z &= 0 \end{aligned}$$

has many solutions. $x = y = z = 0$ is a solution.

'No': For example, the linear system

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 0 \end{aligned}$$

has no solution.

(3) No. Sometimes such a system will have a solution, sometimes it will not.

$\begin{aligned} x &= 0 \\ x + y &= 0 \end{aligned}$ has a solution. But another linear system $\begin{aligned} x + y &= 1 \\ x + y &= 0 \end{aligned}$ does not.

2. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ 2x + 5y + 2z &= -1 \\ y + 3z &= 8 \end{aligned}$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

Solution.

Subtracting the first equation from the second gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ y + 3z &= 8. \end{aligned}$$

Next, subtracting the second equation from the third gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ 5z &= 10, \end{aligned}$$

so $z = 2$. We can back-substitute to find y and then x. The second equation is $y-2z = -2$, so $y-2(2) = -2$, thus $y = 2$. The first equation is $2x+4(2)+4(2) = 1$, so $2x = -15$, thus $x = -15/2$. We have found that the planes intersect at the point

$$\left(-\frac{15}{2},\ 2,\ 2\right).$$

An alternative method would have been to use augmented matrices to isolate z and then back-substitute:

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

The last line is $5z = 10$, so $z = 2$. From here, back-substitution would give us $y = 2$ and then $x = -\frac{15}{2}$, just like before.

3. Find all values of h so that the lines $x + hy = -5$ and $2x - 8y = 6$ do *not* intersect. For all such h, draw the lines $x + hy = -5$ and $2x - 8y = 6$ to verify that they do not intersect.

Solution.

We can use basic algebra, row operations, or geometric intuition.

Using basic algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$\begin{aligned} x + hy &= -5 \\ 2x - 8y &= 6. \end{aligned}$$

Subtracting twice the first equation from the second equation gives us

$$\begin{aligned} x + hy &= -5 \\ (-8-2h)y &= 16. \end{aligned}$$

If $-8-2h=0$ (so $h=-4$), then the second line is $0\cdot y=16$, which is impossible. In other words, if $h=-4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h\neq -4$, then we can solve for y above:

$$(-8-2h)y=16 \qquad y=\frac{16}{-8-2h} \qquad y=\frac{8}{-4-h}.$$

We can now substitute this value of y into the first equation to find x at the point of intersection:

$$x+hy=-5 \qquad x+h\cdot\frac{8}{-4-h}=-5 \qquad x=-5-\frac{8h}{-4-h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h=-4}$.

Using intuition from geometry in $\mathbf{R}^2$: Two non-identical lines in $\mathbf{R}^2$ will fail to intersect, if and only if they are parallel. The second line is $y=\frac{1}{4}x-\frac{3}{4}$, so its slope is $\frac{1}{4}$.

If $h\neq 0$, then the first line is $y=-\frac{1}{h}x-\frac{5}{h}$, so the lines are parallel when $-\frac{1}{h}=\frac{1}{4}$, which means $h=-4$. In this case, the lines are $y=\frac{1}{4}x+\frac{5}{4}$ and $y=\frac{1}{4}x-\frac{3}{4}$, so they are parallel non-intersecting lines.

(If $h=0$ then the first line is vertical and the two lines intersect when $x=-5$).

Using row operations: The problem could be done using augmented matrices, which will soon become our main method for solving systems of equations.

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right).$$

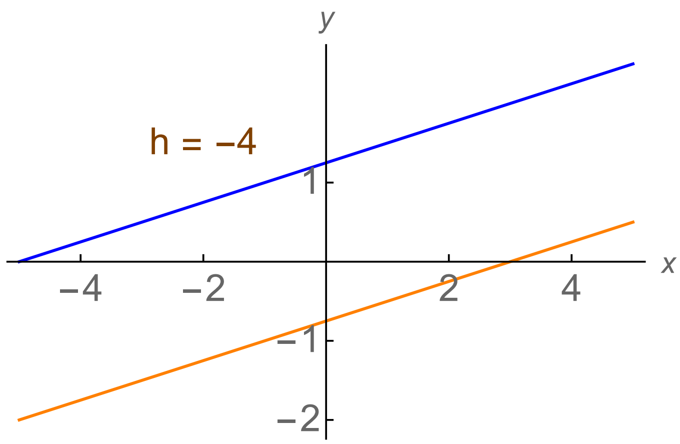
If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h\neq -4$, then the second equation is $(-8-2h)y=16$, so

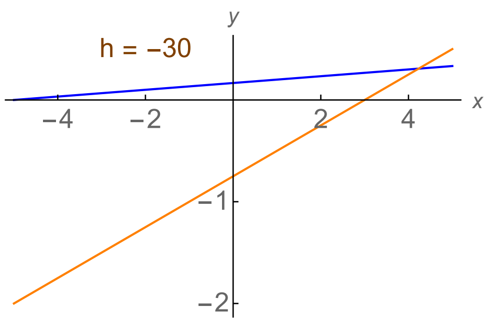
$$y=\frac{16}{-8-2h}=\frac{8}{-4-h} \quad \text{and} \quad x=-5-hy=-5-\frac{8h}{-4-h},$$

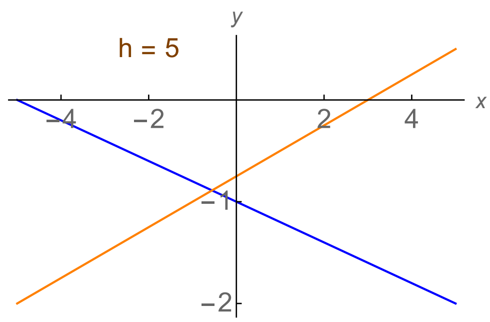
and the lines intersect at (x,y). Therefore, our answer is $h=-4$.

Here are the two lines for $h=-4$, and we can see they are different parallel lines.

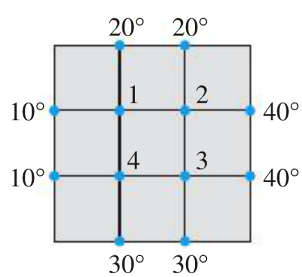


If we vary h away from -4, then the blue and orange lines will have different slopes and will inevitably intersect. For example,





4. The picture below represents the temperatures at four interior nodes of a mesh.



Let $T_1, \dots, T_4$ be the temperatures at nodes 1 through 4. Suppose that the temperature at each node is the average of the four nearest nodes. For example,

$$T_1 = \frac{10 + 20 + T_2 + T_4}{4}.$$

(1) Write a system of four linear equations whose solution would give the temperatures $T_1, \dots, T_4$.
(2) Write an augmented matrix that represents that system of equations.

Solution.

(1) We already have the first equation from above.

$$T_2 = \frac{T_1 + 20 + 40 + T_3}{4}, \qquad \text{or} \qquad -T_1 + 4T_2 - T_3 = 60$$
$$T_3 = \frac{T_4 + T_2 + 40 + 30}{4}, \qquad \text{or} \qquad -T_2 + 4T_3 - T_4 = 70$$
$$T_4 = \frac{10 + T_1 + T_3 + 30}{4}, \qquad \text{or} \qquad -T_1 - T_3 + 4T_4 = 40$$

(2) To put this in matrix form, we arrange the above equations to keep everything in order:

$$\begin{array}{rcrcrcrcl} 4T_1 & - & T_2 & & & - & T_4 & = & 30 \\ -T_1 & + & 4T_2 & - & T_3 & & & = & 60 \\ & & -T_2 & + & 4T_3 & - & T_4 & = & 70 \\ -T_1 & & & - & T_3 & + & 4T_4 & = & 40 \end{array}$$

This gives the augmented matrix

$$\left(\begin{array}{rrrr|r} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{array}\right)$$