Math 1553 Worksheet §2.6, 2.7, 2.9, 3.1, 3.2

Solutions

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×10 matrix with 2 pivots in its RREF, then $\dim(\text{Nul}A) = 8$ and $\text{rank}(A) = 2$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the transformation $T(x) = Ax$ is onto.

TRUE **FALSE**

c) If $\{a, b, c\}$ is a basis of a linear space V, then $\{a, a+b, b+c\}$ is a basis of V as well.

TRUE **FALSE**

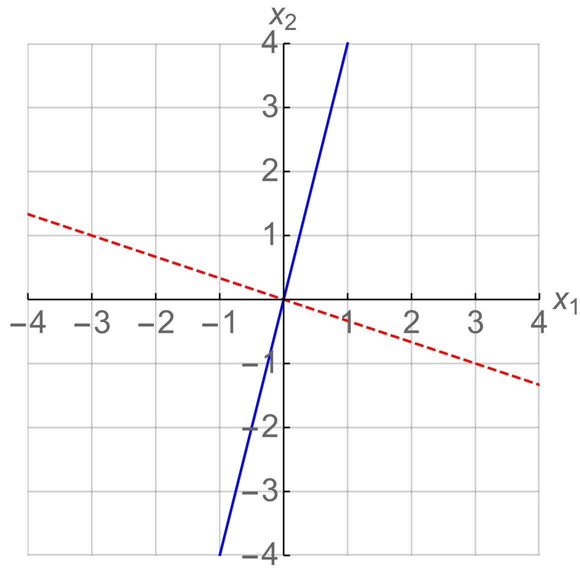
Solution.

a) True. $\text{rank}(A)$ is the same as number of pivots in A. $\dim(\text{Nul}A)$ is the same as the number of free variables. Moreover by the Rank Theorem, $\text{rank}(A) + \dim(\text{Nul}A) = 10$, so $\dim(\text{Nul}A) = 10 - 2 = 8$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. Because a and b are independent, $a + b$ and a are linearly independent, and furthermore a and b are in $\text{Span}\{a, a+b\}$. Next, c is independent from $\{a, b\}$, so $b + c$ is independent from $\{a, a+b\}$, meaning that $\{a, a+b, b+c\}$ is independent by the increasing span criterion. Since $a, a+b, b+c$ are all clearly in $\text{Span}\{a, b, c\}$, by the basis theorem $\{a, a+b, b+c\}$ also form a span for $\text{Span}\{a, b, c\} = V$. Alternatively, we could notice that $a, b, c \in \text{Span}\{a, a+b, b+c\}$, and since $V = \text{Span}\{a, b, c\}$ it is a three-dimensional space spanned by the set of three elements $\{a, a+b, b+c\}$, those three elements must form a basis, by the basis theorem.

2. Write a matrix A so that $\text{Col}(A)$ is the solid blue line and $\text{Nul}(A)$ is the dotted red line drawn below.



Solution.

We'd like to design an A with the prescribed column space $\text{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$ and null space $\text{Span}\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}$.

We start with analyzing the null space. We can write parametric form of the null space:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = t\begin{pmatrix}3\\-1\end{pmatrix} \quad \text{is the same as } \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-3x_2\\x_2\end{pmatrix}$$

Then this implies the RREF of A must be $\begin{pmatrix}1 & 3\\0 & 0\end{pmatrix}$.

Now we need to combine the information that column space is $\text{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$. That means the second row must be 4 multiple of the first row. Therefore the second row must be $\begin{pmatrix}4 & 12\end{pmatrix}$. We conclude,

$$A = \begin{pmatrix}1 & 3\\4 & 12\end{pmatrix}$$

Note any nonzero scalar multiple of the above matrix is also a solution.

3. Let $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$, and let T be the matrix transformation associated to A, so $T(x) = Ax$.

a) What is the domain of T? What is the codomain of T? Give an example of a vector in the range of T.

b) The RREF of A is $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Is there a vector in the codomain of T which is not in the range of T? Justify your answer.

c) Is T one-to-one? Is T onto? Justify your answer.

Solution.

a) The domain is $\mathbf{R}^4$; the codomain is $\mathbf{R}^3$. The vector $0 = T(0)$ is contained in the range, as is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

b) Yes. The range of T is the column span of A, and from the RREF of A we know A only has two pivots, so its column span is a 2-dimensional subspace of $\mathbf{R}^3$. Since $\dim(\mathbf{R}^3) = 3$, the range is not equal to $\mathbf{R}^3$.

c) T is neither one-to-one nor onto. T is not onto since range(T), namely column span of A, is strictly smaller than codomain. T is not one-to-one, since there are infinitely many solutions to $Ax = 0$, which is infinite-to-one.

4. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the transformation is not one-to-one, find two vectors with the same image.

a) Counterclockwise rotation by 32° in $\mathbf{R}^2$.

b) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.

c) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.

d) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.

Solution.

a) This is both one-to-one and onto. If v is any vector in $\mathbf{R}^2$, then there is one and only one vector w such that $T(w) = v$, namely, the vector that is rotated $-32°$ from v.

b) This is onto. If (a,b) is any vector in the codomain $\mathbf{R}^2$, then $(a,b) = T(b,0,a)$, so (a,b) is in the range. It is not one-to-one though: indeed, $T(0,0,0) = (0,0) = T(0,1,0)$. Alternatively, we could have observed that T is a matrix transformation and examined its matrix A: $T(x) = Ax$ for

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since A has a pivot in every row but not every column, T is onto but not one-to-one.

c) This is not onto. There is no (x,y,z) such that $T(x,y,z) = (1,0)$. It is not one-to-one: for instance, $T(0,0,0) = (0,0) = T(0,1,0)$.

d) The transformation T with matrix A is onto if and only if A has a pivot in every *row*, and it is one-to-one if and only if A has a pivot in every *column*. So we row reduce:

$$A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has a pivot in every column, so T is one-to-one. It does not have a pivot in every row, so it is not onto. To find a specific vector b in $\mathbf{R}^3$ which is not in the image of T, we have to find a $b = (b_1, b_2, b_3)$ such that the matrix equation $Ax = b$ is inconsistent. We row reduce again:

$$\left(A \mid b \right) = \left(\begin{array}{cc|c} 1 & 6 & b_1 \\ -1 & 2 & b_2 \\ 2 & -1 & b_3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & \text{don't care} \\ 0 & 1 & \text{don't care} \\ 0 & 0 & -3b_1 + 13b_2 + 8b_3 \end{array}\right).$$

Hence any b_1, b_2, b_3 for which $-3b_1 + 13b_2 + 8b_3 \neq 0$ will make the equation $Ax = b$ inconsistent. For instance, $b = (1,0,0)$ is not in the range of T.