Supplemental problems: §3.3

1. Circle **T** if the statement is always true, and circle **F** otherwise.

a) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear and $T(e_1) = T(e_2)$, then the homogeneous equation $T(x) = 0$ has infinitely many solutions.

b) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

2. Consider $T : \mathbf{R}^3 \to \mathbf{R}^4$ given by

$$T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x).$$

Is T one-to-one? Justify your answer.

3. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (y, y)$.

b) JUST FOR FUN: Consider T : (Smooth functions) $\to$ (Smooth functions) given by $T(f) = f'$ (the derivative of f). Then T is not a transformation from any $\mathbf{R}^n$ to $\mathbf{R}^m$, but it is still *linear* in the sense that for all smooth f and g and all scalars c, we have the following (by properties of differentiation we learned in Calculus 1):

$$T(f + g) = T(f) + T(g) \quad \text{since} \quad (f + g)' = f' + g'$$
$$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

Is T one-to-one?

4. In each case, determine whether T is linear. Briefly justify.

a) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$.

b) $T(x, y) = (y, x^{1/3})$.

c) $T(x, y, z) = 2x - 5z$.

5. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy-plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane.

In the worksheet, we found the matrix for the transformation T caused by the wolf. Geometrically describe the image of the house under T.

Supplemental problems: §3.4

1. Consider $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \\ x-y \end{pmatrix}$$

and $U : \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the xy-plane (forgetting the z-coordinate), then rotating counterclockwise by 90°.

a) Compute the standard matrices A and B for T and U, respectively.

b) Compute the standard matrices for $T \circ U$ and $U \circ T$.

c) Circle all that apply:

$T \circ U$ is: one-to-one onto

$U \circ T$ is: one-to-one onto

2. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by 60°. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

b) Find the standard matrix for the transformation that you circled in (b).

3. Find all matrices B that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

4. Let T and U be the (linear) transformations below:

$T(x_1, x_2, x_3) = (x_3 - x_1,\ x_2 + 4x_3,\ x_1,\ 2x_2 + x_3) \qquad U(x_1, x_2, x_3, x_4) = (x_1 - 2x_2,\ x_1).$

a) Which compositions makes sense (circle all that apply)? $U \circ T$ $\quad T \circ U$

b) Compute the standard matrix for T and for U.

c) Compute the standard matrix for each composition that you circled in (a).

5. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A and B are matrices and the products AB and BA are both defined, then A and B must be square matrices with the same number of rows and columns.

b) If A, B, and C are nonzero 2×2 matrices satisfying $BA = CA$, then $B = C$.

c) Suppose A is an 4×3 matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a 3×3 matrix B which is not the zero matrix and satisfies $AB = 0$.

d) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are one-to-one linear transformations. Then $U \circ T$ is one-to-one. (What if U and T are not necessarily linear?)

6. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.

a) A 3×3 matrix P, which is not the identity matrix or the zero matrix, and satisfies $P^2 = P$.

b) A 2×2 matrix A satisfying $A^2 = I$.

c) A 2×2 matrix A satisfying $A^3 = -I$.