Supplemental problems: §3.3

1. Circle **T** if the statement is always true, and circle **F** otherwise.

a) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear and $T(e_1) = T(e_2)$, then the homogeneous equation $T(x) = 0$ has infinitely many solutions.

b) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

Solution.

a) True. The matrix transformation $T(x) = Ax$ is not one-to-one, so $Ax = 0$ has infinitely many solutions. For example, $e_1 - e_2$ is a non-trivial solution to $Ax = 0$ since $A(e_1 - e_2) = Ae_1 - Ae_2 = 0$.

b) True. Let A be the $m \times n$ standard matrix for T. If T is both one-to-one and onto then T must have a pivot in each column and in each row, which is only possible when A is a square matrix ($m = n$).

2. Consider $T : \mathbf{R}^3 \to \mathbf{R}^4$ given by

$$T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x).$$

Is T one-to-one? Justify your answer.

Solution.

One approach: We form the standard matrix A for T:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We row-reduce A until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A has a pivot in every column, so T is one-to-one.

Alternative approach: T is a linear transformation, so it is one-to-one if and only if the equation $T(x, y, z) = (0, 0, 0)$ has only the trivial solution.
If $T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x) = (0, 0, 0, 0)$ then $x = 0$, and

$$x + z = 0 \implies 0 + z = 0 \implies z = 0, \text{ and finally}$$

$$3x - 4y + z = 0 \implies 0 - 4y + 0 = 0 \implies y = 0,$$

so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, T is one-to-one.

3. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x,y,z) = (y,y)$.

b) JUST FOR FUN: Consider T : (Smooth functions) $\to$ (Smooth functions) given by $T(f) = f'$ (the derivative of f). Then T is not a transformation from any $\mathbf{R}^n$ to $\mathbf{R}^m$, but it is still *linear* in the sense that for all smooth f and g and all scalars c, we have the following (by properties of differentiation we learned in Calculus 1):

$$T(f+g) = T(f) + T(g) \quad \text{since} \quad (f+g)' = f' + g'$$

$$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

Is T one-to-one?

Solution.

a) This is not onto. Everything in the range of T has its first coordinate equal to its second, so there is no (x,y,z) such that $T(x,y,z) = (1,0)$. It is not one-to-one: for instance, $T(0,0,0) = (0,0) = T(0,0,1)$.

b) T is not one-to-one. If T were one-to-one, then for any smooth function b, the equation $T(f) = b$ would have at most one solution. However, note that if f and g are the functions $f(t) = t$ and $g(t) = t - 1$, then f and g are different functions but their derivatives are the same, so $T(f) = T(g)$. Therefore, T is not one-to-one. It is not within the scope of Math 1553. If you find it confusing, feel free to ignore it.

4. In each case, determine whether T is linear. Briefly justify.

a) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$.

b) $T(x,y) = (y, x^{1/3})$.

c) $T(x,y,z) = 2x - 5z$.

Solution.

a) Not linear. $T(0,0) = (0,0,1) \neq (0,0,0)$.

b) Not linear. The $x^{1/3}$ term gives it away. $T(0,2) = (0, 2^{1/3})$ but $2T(0,1) = (0,2)$.

c) Linear. In fact, $T(v) = Av$ where

$$A = \begin{pmatrix} 2 & 0 & -5 \end{pmatrix}.$$

5. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, and $(1,1,1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of $45°$ in a counterclockwise direction about the z-axis (look downward onto the xy-plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane.

In the worksheet, we found the matrix for the transformation T caused by the wolf. Geometrically describe the image of the house under T.

Solution.

Work shows that $T(x) = Ax$, where

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We know the house has been effectively destroyed, but what do its remains look like? To get an idea, let's look at what happens to the vertices.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

This indicates the pyramid has been squashed into a triangle in the xy-plane with vertices $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$. (the point $\begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$ is along the top side of this triangle).

Effectively, the pyramid was rotated and then destroyed, so that its (rotated) base is all that remains.

Supplemental problems: §3.4

1. Consider $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ x - y \end{pmatrix}$$

and $U : \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the xy-plane (forgetting the z-coordinate), then rotating counterclockwise by $90°$.

a) Compute the standard matrices A and B for T and U, respectively.

b) Compute the standard matrices for $T \circ U$ and $U \circ T$.

c) Circle all that apply:

$T \circ U$ is: one-to-one onto

$U \circ T$ is: one-to-one onto

Solution.

a) We plug in the unit coordinate vectors to get

$$A = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} | & | & | \\ U(e_1) & U(e_2) & U(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

b) The standard matrix for $T \circ U$ is

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

The standard matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}.$$

c) Looking at the matrices, we see that $T \circ U$ is not one-to-one or onto, and that $U \circ T$ is one-to-one and onto.

2. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by 60°. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) Which composition makes sense? (Circle one.)

$U \circ T \qquad T \circ U$

b) Find the standard matrix for the transformation that you circled in (b).

Solution.

a) Only $U \circ T$ makes sense, as the codomain of T is $\mathbf{R}^2$, which is the domain of U.

b) The standard matrix for $U \circ T$ is

$$BA = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.$$

3. Find all matrices B that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

Solution.

B must have two rows and two columns for the above to compute, so $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We calculate

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} a-3c & b-3d \\ -3a+5c & -3b+5d \end{pmatrix}.$$

Setting this equal to $\begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$ gives us

$$\left.\begin{aligned} a - 3c &= -3 \\ -3a + 5c &= 1 \end{aligned}\right\} \xrightarrow{\text{solve}} a = 3,\ c = 2$$

and

$$\left.\begin{aligned} b - 3d &= -11 \\ -3b + 5d &= 17 \end{aligned}\right\} \xrightarrow{\text{solve}} b = 1,\ d = 4.$$

Therefore, $B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

4. Let T and U be the (linear) transformations below:

$$T(x_1, x_2, x_3) = (x_3 - x_1,\ x_2 + 4x_3,\ x_1,\ 2x_2 + x_3) \qquad U(x_1, x_2, x_3, x_4) = (x_1 - 2x_2,\ x_1).$$

a) Which compositions makes sense (circle all that apply)? $U \circ T$ $\quad$ $T \circ U$

b) Compute the standard matrix for T and for U.

c) Compute the standard matrix for each composition that you circled in (a).

Solution.

a) $U \circ T$ makes sense, but $T \circ U$ does not.

b) Let A be the standard matrix for T and B be the standard matrix for U.

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

c) The matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -7 \\ -1 & 0 & 1 \end{pmatrix}.$$

5. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A and B are matrices and the products AB and BA are both defined, then A and B must be square matrices with the same number of rows and columns.

b) If A, B, and C are nonzero 2×2 matrices satisfying $BA = CA$, then $B = C$.

c) Suppose A is an 4×3 matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a 3×3 matrix B which is not the zero matrix and satisfies $AB = 0$.

d) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are one-to-one linear transformations. Then $U \circ T$ is one-to-one. (What if U and T are not necessarily linear?)

Solution.

a) False. For example, if A is any 2×3 matrix and B is any 3×2 matrix, then AB and BA are both defined.

b) False. Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $BC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but $B \neq C$.

c) True. If T is not one-to-one then there is a non-zero vector v in $\mathbf{R}^3$ so that

$$Av = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The 3×3 matrix $B = \begin{pmatrix} | & | & | \\ v & v & v \\ | & | & | \end{pmatrix}$ satisfies

$$AB = \begin{pmatrix} | & | & | \\ Av & Av & Av \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

d) True. Recall that a transformation S is one-to-one if $S(x) = S(y)$ implies $x = y$ (the same outputs implies the same inputs). Suppose that $U \circ T(x) = U \circ T(y)$. Then $U(T(x)) = U(T(y))$, so since U is one-to-one, we have $T(x) = T(y)$. Since T is one-to-one, this implies $x = y$. Therefore, $U \circ T$ is one-to-one.

Note that this argument does not use the assumption that U and T are linear transformations.

Alternative: We'll show that $U \circ T(x) = 0$ has only the trivial solution. Let A be the matrix for U and B be the matrix for T, and suppose x is a vector satisfying $(U \circ T)(x) = 0$. In terms of matrix multiplication, this is equivalent to $ABx = 0$. Since U is one-to-one, the only solution to $Av = 0$ is $v = 0$, so $A(Bx) = 0 \implies Bx = 0$.

Since T is one-to-one, we know that $Bx = 0 \implies x = 0$. Therefore, the equation $(U \circ T)(x) = 0$ has only the trivial solution.

6. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.

a) A 3×3 matrix P, which is not the identity matrix or the zero matrix, and satisfies $P^2 = P$.

b) A 2×2 matrix A satisfying $A^2 = I$.

c) A 2×2 matrix A satisfying $A^3 = -I$.

Solution.

a) Take P to be the natural projection onto the xy-plane in $\mathbf{R}^3$, so $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If you apply P to a vector then the result will be within the xy-plane of $\mathbf{R}^3$, so applying P a second time won't change anything, hence $P^2 = P$.

b) Take A to be matrix for reflection across the line $y = x$, so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since A swaps the x and y coordinates, repeating A will swap them back to their original positions, so $AA = I$.

c) Note that $-I$ is the matrix that rotates counterclockwise by 180°, so we need a transformation that will give you counterclockwise rotation by 180° if you do it three times. One such matrix is the rotation matrix for 60° counterclockwise,

$$A = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Another such matrix is $A = -I$.