

Supplemental problems: §2.4, §2.5, and some more 2.3

The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A :

$$\begin{array}{l} \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} \begin{pmatrix} \text{HW} & \text{Q} & \text{M} & \text{F} \\ 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix}$$

- Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3 .
 - In a worksheet, you wrote the matrix equation $Ax = b$ to relate your final grades to your scores. Keeping $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as a general vector, write the augmented matrix $(A | b)$.
 - Row reduce this matrix until you reach reduced row echelon form.
 - Looking at the final matrix in (b), what equation in terms of b_1, b_2, b_3 must be satisfied in order for $Ax = b$ to have a solution?
 - The answer to (c) also defines the span of the columns of A . Describe the span geometrically.
 - Solve the equation in (c) for b_1 . Looking at this equation, is it possible for b_1 to be the largest of b_1, b_2, b_3 ? In other words, is it ever possible for the grade under Scheme 1 to be the highest of the three final course grades? Why or why not? Which scheme would you argue for?
- For each of the following, give an example if it is possible. If it is not possible, justify why there is no such example.
 - A 3×4 matrix A in RREF with 2 pivot columns, so that for some vector b , the system $Ax = b$ has exactly three free variables.
 - A homogeneous linear system with no solution.
 - A 5×3 matrix in RREF such that $Ax = 0$ has a non-trivial solution.
 - If $Ax = b$ is consistent, then the solution set is a span.

- Suppose the solution set of a certain system of linear equations is given by

$$x_1 = 9 + 8x_4, \quad x_2 = -9 - 14x_4, \quad x_3 = 1 + 2x_4, \quad x_4 = x_4 \text{ (} x_4 \text{ free)}.$$

Write the solution set in parametric vector form. Describe the set geometrically.

4. a) What best describes $\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$? Justify your answer.

(I) It is a plane through the origin.

(II) It is three lines through the origin.

(III) It is all of \mathbf{R}^3 .

(IV) It is a plane, plus the line through the origin and the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b) Does $\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \right\} = \mathbf{R}^3$? If yes, justify your answer. If not,

write a vector in \mathbf{R}^3 which is not in $\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \right\}$.

5. Let $A = \begin{pmatrix} 5 & -5 & 10 \\ 3 & -3 & 6 \end{pmatrix}$. Draw the column span of A .

6. Consider the following consistent system of linear equations.

$$x_1 + 2x_2 + 3x_3 + 4x_4 = -2$$

$$3x_1 + 4x_2 + 5x_3 + 6x_4 = -2$$

$$5x_1 + 6x_2 + 7x_3 + 8x_4 = -2$$

a) Find the parametric vector form for the general solution.

b) Find the parametric vector form of the corresponding *homogeneous* equations.

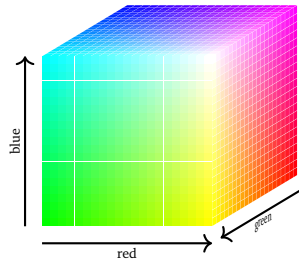
7. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \right\}$ is linearly independent.

b) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ is linearly independent.

c) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly dependent.

8. Every color on my computer monitor is a vector in \mathbf{R}^3 with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors v_1, v_2, \dots, v_p , we can form a “weighted average” of these colors by making a linear combination

$$v = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

with $c_1 + c_2 + \dots + c_p = 1$. Example:

$$\frac{1}{2} \text{ (red square)} + \frac{1}{2} \text{ (blue square)} = \text{ (purple square)}$$

Consider the colors on the right. For which h is

$$\left\{ \begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}, \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix} \right\}$$

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix} \quad \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$

(orange square) (teal square)

linearly dependent? What does that say about the corresponding color?

$h =$

9. Which of the following must be true for any set of seven vectors in \mathbf{R}^5 ? Answer “yes”, “no”, or “maybe” in each case.
- The vectors span \mathbf{R}^5 .
 - The vectors are linearly dependent.
 - At least one of the vectors is in the span of the other six vectors.
 - If we put the seven vectors as the columns of a matrix A , then the matrix equation $Ax = 0$ must have infinitely many solutions.
 - Suppose we put the seven vectors as the columns of a matrix A . Then for each b in \mathbf{R}^5 , the matrix equation $Ax = b$ must be consistent.
 - If every vector b in \mathbf{R}^5 can be written as a linear combination of our seven vectors, then in fact every b in \mathbf{R}^5 can be written in *infinitely many* different ways as a linear combination of our seven vectors.

- 10.** Suppose A is a 2×3 matrix and the solution set to $Ax = 0$ is $\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$. Must it be true that the equation $Ax = b$ is consistent for each b in \mathbf{R}^2 ?
- 12.** Write vectors u , v , and w in \mathbf{R}^4 so that $\{u, v, w\}$ is linearly dependent, but u is not in $\text{Span}\{v, w\}$.