MATH 1553, SPRING 2020
SAMPLE MIDTERM 1A: COVERS THROUGH SECTION 2.5

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are doing so.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

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Problem 1.

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer.

a) **T** **F** The matrix $\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ is in reduced row echelon form.

b) **T** **F** A system of 4 linear equations in 5 variables can have exactly one solution.

c) **T** **F** The vector equation $x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is consistent.

d) **T** **F** Suppose A is an 4×3 matrix whose first column is the sum of its second and third columns. Then the equation $Ax = 0$ has infinitely many solutions.

e) **T** **F** If A is an $m \times n$ matrix and $m > n$, then then there is at least one vector b in $\mathbf{R}^m$ which is not in the span of the columns of A.

Solution.

a) True.

b) False. The augmented matrix can have at most 4 pivots, so there will be at least one free variable, thus any associated system will either be inconsistent or have infinitely many solutfions.

c) False. The associated augmented matrix has a pivot in the rightmost column.

d) True. The columns of A are linearly dependent, thus $Ax = 0$ has infinitely many solutions.

e) True. The matrix A has at most n pivots, but it has m rows and $m > n$ so it cannot have a pivot in every row.

Extra space for scratch work on problem 1

Problem 2.

Short answer. You do not need to show your work or justify your answer.

a) Complete the following mathematical definition of linear independent (be mathematically precise!):
Let $v_1, v_2 \dots, v_p$ be vectors in $\mathbf{R}^n$. We say $\{v_1, \dots, v_p\}$ is *linearly independent* if...

b) Are there three nonzero vectors $v_1,\ v_2,\ v_3$ in $\mathbf{R}^3$ so that $\mathrm{Span}\{v_1,\ v_2,\ v_3\}$ is a plane but v_3 is not in $\mathrm{Span}\{v_1, v_2\}$? If your answer is yes, write such vectors $v_1,\ v_2,\ v_3$ and label each vector clearly.

c) Write a matrix A with the property that the equation $Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is consistent.

d) Suppose A is a 2×3 matrix and v is some vector so that the set of solutions to $Ax = v$ has parametric form

$$x_1 = 1 + x_3 \qquad x_2 = 2 - x_3 \qquad x_3 = x_3 \qquad (x_3 \text{ free}).$$

Which of the following must be true? Circle all that apply.

(i) The solution set for $Ax = 0$ is $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$.

(ii) For each b in $\mathbf{R}^2$, the equation $Ax = b$ is consistent.

(iii) v is not the zero vector.

Solution.

a) ...the equation $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$ has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

b) For example, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

c) All we need is a matrix with 3 rows, so that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is in the span of its columns.

For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or even} \quad A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or } \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) We see (i) is true because the homogeneous solution set is the translation of a consistent set by any particular solution (here $(1, 2, 0)$ is a particular solution)

We see (ii) is true because A must have two pivots for the equation $Ax = v$ to have exactly one free variable, which means A has a pivot in every row.

We see (iii) is true because if $v = 0$ then $x = 0$ would be a solution to $Ax = v$, but the solution set to $Ax = v$ doesn't include the origin here (if it did then $x_3 = 0$ but then $x_1 = 1$ and $x_2 = 2$).

Extra space for work on problem 2

Problem 3.

Parts (a) and (b) are unrelated.

a) John Dioguardi cannot stop thinking about the system of equations

$$x - 4y = h$$
$$-3x + ky = 4,$$

where h and k are real numbers.

For what values of h and k (if any) is the system inconsistent?

b) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}, v_3 = \begin{pmatrix} -3 \\ 0 \\ -9 \end{pmatrix}$. Are the vectors $v_1,\ v_2,\ v_3$ linearly independent or linearly dependent? If they are linearly independent, justify why. If they are linearly dependent, write one vector as a linear combination of the other vectors.

Solution.

a) We do one step of row-reduction:

$$\left(\begin{array}{cc|c} 1 & -4 & h \\ -3 & k & 4 \end{array}\right) \xrightarrow{R_2 = R_2 + 3R_1} \left(\begin{array}{cc|c} 1 & -4 & h \\ 0 & k-12 & 4+3h \end{array}\right).$$

The system will be consistent unless $k - 12 = 0$ and $4 + 3h \neq 0$. Thus,

$$k = 12 \quad \text{and} \quad h \neq -\frac{4}{3}.$$

b) We put the vectors as columns of a matrix and row reduce:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -3 & -6 & -9 \end{pmatrix} \xrightarrow[R_3 = R_3 + 3R_1]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 0 & -18 \end{pmatrix}.$$

The matrix has a pivot in every column, so the vectors are linearly independent.

Extra space for work on problem 3

Problem 4.

Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3$:

$$\begin{gathered} x_1 - 2x_2 + 2x_3 = 1 \\ 5x_1 - 10x_2 + 12x_3 = -3 \\ -3x_1 + 6x_2 - 6x_3 = -3. \\ 2x_1 - 4x_2 + 5x_3 = -2. \end{gathered}$$

a) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

b) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

c) Write *one* specific vector that solves the system of equations.

Solution.

a)

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 5 & -10 & 12 & -3 \\ -3 & 6 & -6 & -3 \\ 2 & -4 & 5 & -2 \end{array}\right) \xrightarrow[R_3=R_3+3R_1,\, R_4=R_4-2R_1]{R_2=R_2-5R_1} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow[\text{then destroy } R_4]{R_2=R_2/2} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-2R_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 9 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

b) From (a) we see x_2 is free, and

$$x_1 = 9 + 2x_2, \qquad x_2 = x_2, \qquad x_3 = -4.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 + 2x_2 \\ x_2 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ -4 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

c) Many examples possible. For example, $\begin{pmatrix} 9 \\ 0 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 11 \\ 1 \\ -4 \end{pmatrix}$.

Extra space for work on problem 4

Problem 5.

Parts (a) and (b) are unrelated.

a) Write an augmented matrix in RREF representing a system of three equations in two unknowns, whose solution set is the line $y = 2x$ in $\mathbf{R}^2$.

b) Let $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$. Draw the span of the columns of A below.

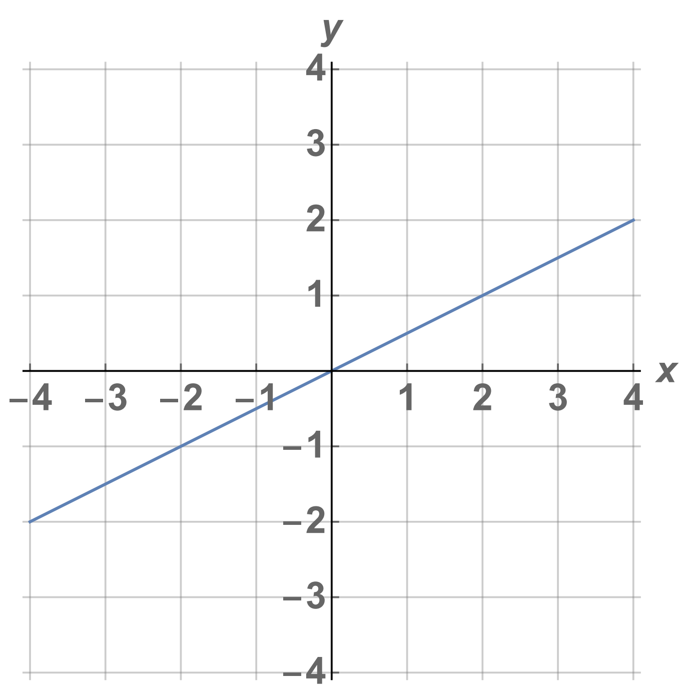
Solution.

a) We need the left side of the augment to be 3×2. Since the solution set includes the origin, the right side of the augment must be the zero vector.

Now $y = 2x$ is the equation $x = \frac{y}{2}$, so the first row is $\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \end{array}\right)$.

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

b) The first and second columns are both scalar multiples of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so the column span is just the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is the line through the origin and $(2, 1)$.



Extra space for work on problem 5