Supplemental problems: §2.3, §2.4

Problems 1 and 2 use the same widgets and gizmos class from a worksheet. The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{r} \\ \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

1. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Determine which grading scheme leaves you with the highest overall course grade.

Solution.

According to equation (*) above, your final grades would be

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} .90 \\ .90 \\ .85 \\ .95 \end{pmatrix} = \begin{pmatrix} .89 \\ .90 \\ .88 \end{pmatrix}.$$

Hence the second grading scheme gives you the best final grade.

2. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3.

a) In a worksheet, you wrote the matrix equation $Ax = b$ to relate your final grades to your scores. Keeping $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as a general vector, write the augmented matrix $(A \mid b)$.

b) Row reduce this matrix until you reach reduced row echelon form.

c) Looking at the final matrix in (b), what equation in terms of b_1, b_2, b_3 must be satisfied in order for $Ax = b$ to have a solution?

d) The answer to (c) also defines the span of the columns of A. Describe the span geometrically.

e) Solve the equation in (c) for b_1. Looking at this equation, is it possible for b_1 to be the largest of b_1, b_2, b_3? In other words, is it ever possible for the grade under Scheme 1 to be the highest of the three final course grades? Why or why not? Which scheme would you argue for?

Solution.

a) $\left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0.1 & 0.1 & 0.4 & 0.4 & b_2 \\ 0.1 & 0.1 & 0.6 & 0.2 & b_3 \end{array}\right)$

b) Here is the row reduction:

$$\left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0.1 & 0.1 & 0.4 & 0.4 & b_2 \\ 0.1 & 0.1 & 0.6 & 0.2 & b_3 \end{array}\right) \xrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} \left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0 & 0 & -0.1 & 0.1 & b_2-b_1 \\ 0 & 0 & 0.1 & -0.1 & b_3-b_1 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0 & 0 & -0.1 & 0.1 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{array}\right)$$

$$\xrightarrow[R_2=R_2\times(-10)]{R_1=R_1\times 10} \left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 10b_1 \\ 0 & 0 & 1 & -1 & 10b_1-10b_2 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-5R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 8 & -40b_1+50b_2 \\ 0 & 0 & 1 & -1 & 10b_1-10b_2 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{array}\right)$$

c) The last row in the row-reduced matrix translates into $0 = b_2+b_3-2b_1$. Hence the system of equations is inconsistent unless $\boxed{b_2+b_3-2b_1=0}$.

d) This is the plane in $\mathbf{R}^3$ given by $-2b_1+b_2+b_3=0$.

e) Rearranging, this is the set of points (b_1, b_2, b_3) where $b_1 = \frac{1}{2}(b_2+b_3)$, i.e., where b_1 is the average of b_2 and b_3. Hence it is impossible for b_1 to be larger than both b_2 and b_3.

You should argue for the second grading scheme if your final grade was higher than your midterm grade; otherwise you should argue for the third.

3. True or false. If the statement is *ever* false, answer false. Justify your answer.

a) A matrix equation $Ax = b$ is consistent if A has a pivot in every column.

b) If an $m \times n$ matrix A has fewer than n pivots and b is in $\mathbf{R}^m$, then $Ax = b$ has infinitely many solutions.

c) Suppose A is a 3×3 matrix and there is a vector y in $\mathbf{R}^3$ so that $Ax = y$ does not have a solution. Is it possible that there is a z in $\mathbf{R}^3$ so that the equation $Ax = z$ has a *unique* solution? Justify your answer.

d) There is a matrix A and a nonzero vector b so that the solution set of $Ax = b$ is a plane through the origin.

e) Suppose A is an $m \times n$ matrix and b is in $\mathbf{R}^m$. If the columns of A span $\mathbf{R}^m$, then $Ax = b$ must be consistent.

f) If $Ax = b$ is consistent, then the solution set is a span.

Solution.

a) False. For example, the system $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution, even though the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a pivot in every column. However, the system is guaranteed to be consistent if A has a pivot in every **row**.

b) False: For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has an A with one pivot but has no solutions.

c) False. Since $Ax = y$ is inconsistent for some y in $\mathbf{R}^3$, it follows that A has at least one row without a pivot, so A has at most 2 pivots. Therefore, at least one of the three columns of A will not have a pivot, so if an equation $Ax = z$ is consistent, the system will have a free variable and thus infinitely many solutions.

d) False. If the solution set to $Ax = b$ is a plane through the origin, then $x = 0$ is a solution, so $b = A(0)$ and therefore $b = 0$.

e) True. The span of the columns of A is exactly the set of all v for which $Ax = v$ is consistent. Since the span is $\mathbf{R}^m$, the matrix equation is consistent no matter what b is.

f) False. It is a *translate* of a span (unless $b = 0$).

4. For each of the following, give an example if it is possible. If it is not possible, justify why there is no such example.

a) A 3×4 matrix A in RREF with 2 pivot columns, so that for some vector b, the system $Ax = b$ has exactly three free variables.

b) A homogeneous linear system with no solution.

c) A 5×3 matrix in RREF such that $Ax = 0$ has a non-trivial solution.

Solution.

a) Not possible. If A had 2 pivot columns and 3 free variables then it would have 5 columns.

b) Not possible. Any homogeneous linear system has the trivial solution.

c) Yes. For the matrix A below, the system $Ax = 0$ will have two free variables and thus infinitely many solutions.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Suppose the solution set of a certain system of linear equations is given by

$$x_1 = 9 + 8x_4, \qquad x_2 = -9 - 14x_4, \qquad x_3 = 1 + 2x_4, \qquad x_4 = x_4 \ \ (x_4 \text{ free}).$$

Write the solution set in parametric vector form. Describe the set geometrically.

Solution.

In parametric vector form, the solutions are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 + 8x_4 \\ -9 - 14x_4 \\ 1 + 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ -9 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 8 \\ -14 \\ 2 \\ 1 \end{pmatrix}.$$

This is the line in $\mathbf{R}^4$ through $\begin{pmatrix} 9 \\ -9 \\ 1 \\ 0 \end{pmatrix}$ parallel to $\text{Span}\left\{\begin{pmatrix} 8 \\ -14 \\ 2 \\ 1 \end{pmatrix}\right\}$.

6. **a)** What best describes $\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$? Justify your answer.

(I) It is a plane through the origin.

(II) It is three lines through the origin.

(III) It is all of $\mathbf{R}^3$.

(IV) It is a plane, plus the line through the origin and the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b) Does $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}\right\} = \mathbf{R}^3$? If yes, justify your answer. If not, write a vector in $\mathbf{R}^3$ which is not in the span of those three vectors.

Solution.

a) It is all of $\mathbf{R}^3$. From the RREF in part (a), we know that the matrix $\begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has a pivot in every row, so its columns span $\mathbf{R}^3$.

b) No. The first and third vectors are scalar multiples of each other, so we can see the three vectors cannot span $\mathbf{R}^3$. Note that any vector in the span has first coordinate equal to the negative of the third coordinate, so (for example) $\begin{pmatrix}1\\0\\2\end{pmatrix}$ is not in the span.

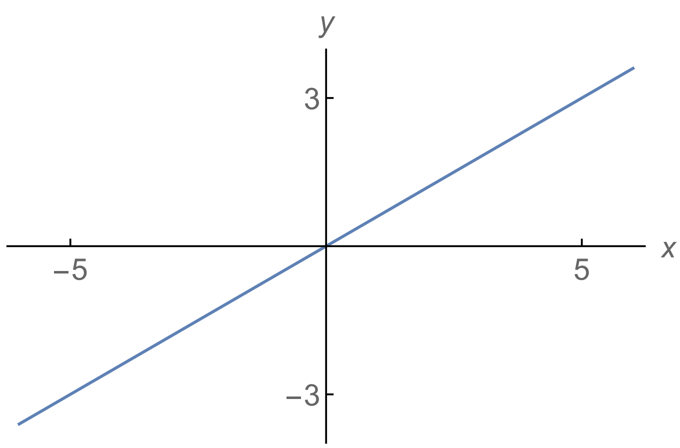
7. Let $A = \begin{pmatrix}5 & -5 & 10\\3 & -3 & 6\end{pmatrix}$. Draw the column span of A.

Solution.

Let v_1, v_2, v_3 be the columns of A. The columns are scalar multiples of each other: $v_2 = -v_1$ and $v_3 = 2v_1$. This means that all three vectors are on the same line through the origin, so

$$\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1\} = \text{Span}\left\{\begin{pmatrix}5\\3\end{pmatrix}\right\}.$$

This is the line through the origin and $\begin{pmatrix}5\\3\end{pmatrix}$, namely the line $y = \dfrac{3x}{5}$.



8. Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2\\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2\\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) Find the parametric vector form for the general solution.

b) Find the parametric vector form of the corresponding *homogeneous* equations.

Solution.

a) We put the equations into an augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 3 & 4 & 5 & 6 & -2 \\ 5 & 6 & 7 & 8 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & -2 & -4 & -6 & 4 \\ 0 & -4 & -8 & -12 & 8 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This means x_3 and x_4 are free, and the general solution is

$$\begin{cases} x_1 \qquad - \ x_3 - 2x_4 = 2 \\ \qquad x_2 + 2x_3 + 3x_4 = -2 \end{cases} \implies \begin{cases} x_1 = x_3 + 2x_4 + 2 \\ x_2 = -2x_3 - 3x_4 - 2 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

This gives the parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

b) Part (a) shows that the solution set of the original equations is the translate of

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{by} \quad \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

We know that the solution set of the homogeneous equations is the parallel plane through the origin, so it is

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Hence the parametric vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Supplemental problems: §2.5

1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\0\\\pi\end{pmatrix}, \begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix}\right\}$ is linearly independent.

b) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ is linearly independent.

c) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is linearly dependent.

Solution.

a) You can eyeball linear independence: if

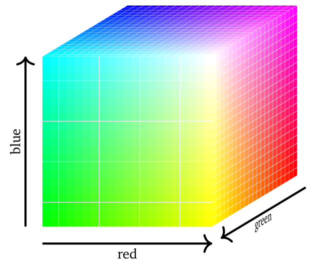
$$x\begin{pmatrix}3\\3\\4\end{pmatrix} + y\begin{pmatrix}0\\0\\\pi\end{pmatrix} + z\begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}3x\\3x+y\sqrt{2}\\4x+\pi z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

then $x = 0$, so $y = z = 0$ too.

b) Since the first coordinate of $\begin{pmatrix}3\\3\\4\end{pmatrix}$ is nonzero, $\begin{pmatrix}3\\3\\4\end{pmatrix}$ cannot be in the span of $\left\{\begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$. And $\begin{pmatrix}0\\10\\20\end{pmatrix}$ is not in the span of $\left\{\begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ because it is not a multiple. Hence the span gets bigger each time you add a vector, so they're linearly independent.

c) Any four vectors in $\mathbf{R}^3$ are linearly dependent; you don't need row reduction for that.

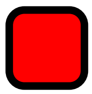
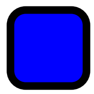
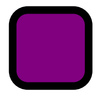
2. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

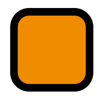
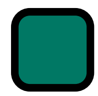
$\frac{1}{2}$  $+$ $\frac{1}{2}$  $=$ 

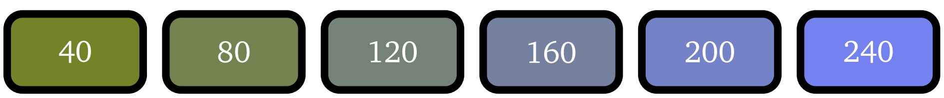
Consider the colors on the right. For which h is

$$\left\{ \begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}, \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix} \right\}$$

linearly dependent? What does that say about the corresponding color?

$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}$ $\begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$

$h =$ 

Solution.

The vectors

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \quad \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}, \quad \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix}$$

are linearly dependent if and only if the vector equation

$$x \begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix} + y \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix} + z \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nonzero solution. This translates into the matrix

$$\begin{pmatrix} 180 & 100 & 116 \\ 50 & 150 & 130 \\ 200 & 100 & h \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & .2 \\ 0 & 1 & .8 \\ 0 & 0 & h-120 \end{pmatrix},$$

which has a free variable if and only if $\boxed{h = 120}$.

Suppose now that $h = 120$. The parametric form for the solution the above vector equation is

$$x = -.2z$$
$$y = -.8z.$$

Taking $z = 1$ gives the linear combination

$$-.2\begin{pmatrix}180\\50\\200\end{pmatrix} - .8\begin{pmatrix}100\\150\\100\end{pmatrix} + \begin{pmatrix}116\\130\\120\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

In terms of colors:

$$\begin{pmatrix}116\\130\\120\end{pmatrix} = \frac{1}{5}\begin{pmatrix}180\\50\\200\end{pmatrix} + \frac{4}{5}\begin{pmatrix}100\\150\\100\end{pmatrix} = \begin{pmatrix}36\\10\\40\end{pmatrix} + \begin{pmatrix}80\\120\\80\end{pmatrix}$$