Section 3.5 and 3.6

Matrix Inverses and the Invertible Matrix Theorem

The Definition of Inverse

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

✔

Poll

Do there exist two matrices A and B such that AB is the identity, but BA is not? If so, find an example. (Both products have to make sense.)

Yes, for instance: $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

However

If A and B are *square* matrices, then

$$AB = I_n \quad \text{if and only if} \quad BA = I_n.$$

So in this case you only have to check one.

Theorem

If A is invertible, then $Ax = b$ has exactly one solution for every b, namely:

$$x = A^{-1}b.$$

Why? Divide by A!

$$Ax = b \rightsquigarrow A^{-1}(Ax) = A^{-1}b \rightsquigarrow (A^{-1}A)x = A^{-1}b$$
$$\rightsquigarrow I_n x = A^{-1}b \rightsquigarrow x = A^{-1}b.$$

$I_n x = x$ for every x

Important

If A is invertible and you know its inverse, then the easiest way to solve $Ax = b$ is by "dividing by A":

$$x = A^{-1}b.$$

This is very convenient when you have to vary b!

Example

Solve the system

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 \phantom{{}+3x_2} + 3x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned} \qquad \text{using} \qquad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Answer: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$

The advantage of using inverses is it doesn't matter what's on the right-hand side of the $=$:

$$\left\{ \begin{aligned} 2x_1 + 3x_2 + 2x_3 &= b_1 \\ x_1 \phantom{{}+3x_2} + 3x_3 &= b_2 \\ 2x_1 + 2x_2 + 3x_3 &= b_3 \end{aligned} \right. \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}.$$

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = \cancel{A^{-1}B^{-1}}\ B^{-1}A^{-1}$.

   Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

i. $A^{-1}B^{-1}C^{-1}$ ii. $B^{-1}A^{-1}C^{-1}$ $\boxed{\text{iii. } C^{-1}B^{-1}A^{-1}}$ iv. $C^{-1}A^{-1}B^{-1}$

Check:

$$\begin{aligned}(ABC)(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1} \\ &= AA^{-1} = I_n.\end{aligned}$$

In general, a product of invertible matrices is invertible, and the inverse is the product of the inverses, in the *reverse order*.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\boxed{\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.}$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $\boxed{A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.}$
2. If $\det(A) = 0$, then A is not invertible.

Why 1?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So we get the identity by dividing by $ad - bc$.

Example

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Computing A^{-1}

In general

Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

1. Row reduce the augmented matrix $(A \mid I_n)$.
2. If the result has the form $(I_n \mid B)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

[interactive]

Computing A^{-1}

Example

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 \div 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

So $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}$.

Check: $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✔

Why Does This Work?

We can think of the algorithm as simultaneously solving the equations

$$Ax_1 = e_1 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_2 = e_2 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_3 = e_3 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

Now note $A^{-1}e_i = A^{-1}(Ax_i) = x_i$, and x_i is the ith column in the augmented part. Also $A^{-1}e_i$ is the ith column of A^{-1}.

Invertible Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

Fact

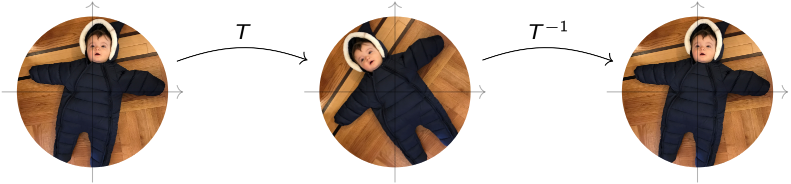
A transformation T is invertible if and only if it is both one-to-one and onto.

If T is one-to-one and onto, this means for every y in $\mathbf{R}^n$, there is a unique x in $\mathbf{R}^n$ such that $T(x) = y$. Then $T^{-1}(y) = x$.

Invertible Transformations

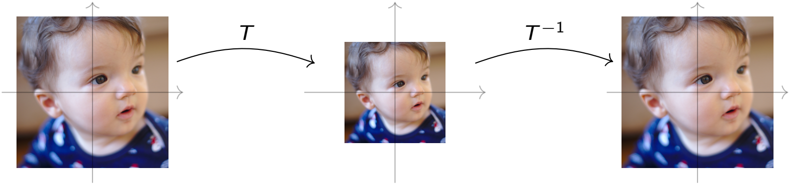
Examples

Let T = counterclockwise rotation in the plane by 45°. What is T^{-1}?



T^{-1} is *clockwise* rotation by 45°. [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let T = shrinking by a factor of $2/3$ in the plane. What is T^{-1}?



T^{-1} is *stretching* by $3/2$. [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let T = projection onto the x-axis. What is T^{-1}? It is not invertible: you can't undo it.

Invertible Linear Transformations

If $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix A, then what is the matrix for T^{-1}?

Let B be the matrix for T^{-1}. We know $T \circ T^{-1}$ has matrix AB, so for all x,

$$ABx = T \circ T^{-1}(x) = x.$$

Hence $AB = I_n$, so $B = A^{-1}$.

Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Invertible Linear Transformations

Examples

Let T = counterclockwise rotation in the plane by 45°. Its matrix is

$$A = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by -45°. Its matrix is

$$B = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Check: $$AB = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

Let T = shrinking by a factor of $2/3$ in the plane. Its matrix is

$$A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Then T^{-1} = stretching by $3/2$. Its matrix is

$$B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

Check: $$AB = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\mathrm{Nul}(A) = \{0\}$.
7. $\mathrm{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\mathrm{Col}\, A = \mathbf{R}^n$.
15. $\dim \mathrm{Col}\, A = n$.
16. $\mathrm{rank}\, A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and try to figure out on your own (or at least with help from the book) why they're all equivalent.

You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.

Question: Is this matrix invertible?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

The second column is a multiple of the first, so the columns are linearly dependent.

A does not satisfy condition (8) of the IMT, so it is not invertible.

Problem: Let A be a 3×3 matrix such that

$$A \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

Show that the rank of A is at most 2.

If we set

$$b = A \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix},$$

then $Ax = b$ has multiple solutions, so it does not satisfy condition (12) of the IMT.

Hence it also does not satisfy condition (16), so the rank is not 3.

In any case the rank is at most 3, so it must be less than 3.

Summary

- The **inverse** of a square matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).
- If A is invertible, then you can solve $Ax = b$ by "dividing by A": $b = A^{-1}x$. There is a unique solution $x = A^{-1}b$ for every x.
- You compute A^{-1} (and whether A is invertible) by row reducing $\left(\begin{array}{c|c} A & I_n \end{array}\right)$. There's a trick for computing the inverse of a 2×2 matrix in terms of determinants.
- A linear transformation T is invertible if and only if its matrix A is invertible, in which case A^{-1} is the matrix for T^{-1}.
- The Invertible Matrix theorem is a list of a zillion equivalent conditions for invertibility that you have to learn (and should understand, since it's well within what we've covered in class so far).