Math 1553 Worksheet: Chapter 5 and 6.1

1. Let $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$.

a) Compute $\det(A)$ using row reduction.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

Solution.

a) Below, r counts the row swaps and s measures the scaling factors.

$$\begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \quad (r = 0, \ s = \frac{1}{2})$$

$$\xrightarrow[R_3 = R_3 + 3R_1, \ R_4 = R_4 - R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0, \ s = \frac{1}{2})$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0, \ s = \frac{1}{2})$$

$$\xrightarrow{R_4 = R_4 - \frac{R_2}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (r = 0, \ s = \frac{1}{2})$$

$$\det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{36}$.

c) $\det(A^T) = \det(A) = -36$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-36)^5 = -60,466,176$$

2. Play matrix tic-tac-toe!

Instead of X against O, we have 1 against 0. The 1-player wins if the final matrix has nonzero determinant, while the 0-player wins if the determinant is zero. You can change who goes first, and you can also modify the size of the matrix.

Click the link above, or copy and paste the url below:

http://textbooks.math.gatech.edu/ila/demos/tictactoe/tictactoe.html

Can you think of a winning strategy for the 0 player who goes first in the 2×2 case? Is there a winning strategy for the 1 player if they go first in the 2×2 case?

3. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and justify why it is false. In every case, assume that A is an $n \times n$ matrix.

a) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

b) If v_1 and v_2 are linearly independent eigenvectors of A, then they must correspond to different eigenvalues.

Solution.

a) False. The RREF of A gives us almost no info about eigenvalues or eigenvectors. To get the eigenvectors corresponding to an eigenvalue λ, we put $A - \lambda I$ into RREF and write the solutions of $\left(A - \lambda I \mid 0 \right)$ in parametric vector form.

b) False. For example, if $A = I_2$ then e_1 and e_2 are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

4. In what follows, T is a linear transformation with matrix A. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

a) $T =$ projection onto the xz-plane in $\mathbf{R}^3$.

b) $T =$ reflection over $y = 2x$ in $\mathbf{R}^2$.

Solution.

a) $T(x, y, z) = (x, 0, z)$, so T fixes every vector in the xz-plane and destroys every vector of the form $(0, a, 0)$ with a real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$.

The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of x and z is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$. In other words:

The 1-eigenspace consists of all vectors in Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, while the 0-eigenspace consists of all vectors in Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

b) T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2, 1) = (2, -1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.