Math 1553, Extra Practice for Midterm 2 (sections 3.6-4.5)

1. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace V of $\mathbf{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

b) **T** **F** If A is an $n \times n$ matrix and $Ae_1 = Ae_2$, then the homogeneous equation $Ax = 0$ has infinitely many solutions.

c) **T** **F** The solution set of a consistent matrix equation $Ax = b$ is a subspace.

d) **T** **F** There exists a 3×5 matrix with rank 4.

e) **T** **F** If A is an 9×4 matrix with a pivot in each column, then

$$\mathrm{Nul} A = \{0\}.$$

f) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^\mathbf{m}$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

g) **T** **F** A translate of a span is a subspace.

h) **T** **F** There exists a 4×7 matrix A such that $\mathrm{nullity} A = 5$.

i) **T** **F** If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

2. Short answer questions: you need not explain your answers.

a) Write a nonzero vector in Col$A$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

b) Complete the following definition:

A transformation $T : \mathbf{R}^n \to \mathbf{R}^{\mathbf{m}}$ *is one-to-one if...*

c) Which of the following are onto transformations? (Check all that apply.)

- ☐ $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane
- ☐ $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane
- ☐ $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate
- ☐ $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

d) Let A be a square matrix and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

- ☐ T is one-to-one
- ☐ $Ax = 0$ is consistent
- ☐ Col$A = \mathbf{R}^n$
- ☐ There is a transformation U such that $T \circ U(x) = x$ for all x

3. Parts (a) and (b) are unrelated.

a) Consider $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x).$$

Is T one-to-one? Justify your answer.

b) Find all real numbers h so that the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

4. Determine which of the following transformations are linear.

a) $S : \mathbf{R}^2 \to \mathbf{R}^2$ given by $S(x_1, x_2) = (x_1,\ 3 + x_2)$

b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1 - x_2,\ x_1 x_2)$

c) $U : \mathbf{R}^2 \to \mathbf{R}^3$ given by $U(x_1, x_2) = (-x_2,\ x_1, 0)$

5. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by $60°$. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

b) Find the standard matrix for the transformation that you circled in (b).

6. Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \\ 5 & 10 & 6 & -17 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for $\mathrm{Col}A$.

b) Find a basis $\mathcal{B}$ for $\mathrm{Nul}A$.

c) For each of the following vectors v, decide if v is in $\mathrm{Nul}A$, and if so, write x as a linear combination of your basis from part (b).

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix}$$

7. Consider $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ x - y \end{pmatrix}$$

and $U : \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the xy-plane (forgetting the z-coordinate), then rotating counterclockwise by $90°$.

a) Compute the standard matrices A and B for T and U, respectively.

b) Compute the standard matrices for $T \circ U$ and $U \circ T$.

c) Circle all that apply:

$T \circ U$ is: one-to-one onto

$U \circ T$ is: one-to-one onto

8. **a)** Write a 2×2 matrix A with **rank** 2, and draw pictures of Nul$A$ and Col$A$.

$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ Nul $A =$ Col $A =$

b) Write a 2×2 matrix B with **rank** 1, and draw pictures of Nul$B$ and Col$B$.

$B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ Nul $B =$ Col $B =$

c) Write a 2×2 matrix C with **rank** 0, and draw pictures of Nul$C$ and Col$C$.

$C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ Nul $C =$ Col $C =$

(In the grids, the dot is the origin.)

9. Suppose A is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \qquad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \qquad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find A.