Math 1553, Extra Practice for Midterm 2 (sections 3.6-4.5)

Solutions

1. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace V of $\mathbf{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

b) **T** **F** If A is an $n \times n$ matrix and $Ae_1 = Ae_2$, then the homogeneous equation $Ax = 0$ has infinitely many solutions.

c) **T** **F** The solution set of a consistent matrix equation $Ax = b$ is a subspace.

d) **T** **F** There exists a 3×5 matrix with rank 4.

e) **T** **F** If A is an 9×4 matrix with a pivot in each column, then

$$\mathrm{Nul} A = \{0\}.$$

f) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^{\mathbf{m}}$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

g) **T** **F** A translate of a span is a subspace.

h) **T** **F** There exists a 4×7 matrix A such that $\mathrm{nullity} A = 5$.

i) **T** **F** If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

Solution.

a) True: if $\{v_1, v_2, v_3\}$ is linearly dependent then $\{v_1, v_2, v_3, v_4\}$ is automatically linearly dependent, which is impossible since $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace.

b) True: The matrix transformation $T(x) = Ax$ is not one-to-one, so $Ax = 0$ has infinitely many solutions. For example, $e_1 - e_2$ is a non-trivial solution to $Ax = 0$ since $A(e_1 - e_2) = Ae_1 - Ae_2 = 0$.

c) False: this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.

d) False: the rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

e) True.

f) True. Let A be the $m \times n$ standard matrix for T. If T is both one-to-one and onto then T must have a pivot in each column and in each row, which is only possible when A is a square matrix ($m = n$).

g) False. A subspace must contain 0.

h) True. For instance,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

i) True. Any basis of $\mathbf{R}^4$ has 4 vectors.

2. Short answer questions: you need not explain your answers.

a) Write a nonzero vector in Col$A$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution.

Either column will work. For instance, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

b) Complete the following definition:

A transformation $T : \mathbf{R}^n \to \mathbf{R}^{\mathrm{m}}$ *is one-to-one if...*

...for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.

c) Which of the following are onto transformations? (Check all that apply.)

- [x] $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane
- [ ] $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane
- [x] $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate
- [x] $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

d) Let A be a square matrix and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

- [x] T is one-to-one
- [ ] $Ax = 0$ is consistent
- [x] Col$A = \mathbf{R}^n$
- [x] There is a transformation U such that $T \circ U(x) = x$ for all x

3. Parts (a) and (b) are unrelated.

a) Consider $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T(x,y,z) = (x,\ x+z,\ 3x-4y+z,\ x).$$

Is T one-to-one? Justify your answer.

b) Find all real numbers h so that the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

Solution.

a) One approach: We form the standard matrix A for T:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We row-reduce A until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[R_3=R_3-3R_1,\ R_4=R_4-R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A has a pivot in every column, so T is one-to-one.

Alternative approach: T is a linear transformation, so it is one-to-one if and only if the equation $T(x,y,z) = (0,0,0)$ has only the trivial solution.
If $T(x,y,z) = (x,\ x+z,\ 3x-4y+z,\ x) = (0,0,0,0)$ then $x = 0$, and

$$x+z = 0 \implies 0+z = 0 \implies z = 0, \text{ and finally}$$

$$3x-4y+z = 0 \implies 0-4y+0 = 0 \implies y = 0,$$

so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, T is one-to-one.

b) We row-reduce A to find when it will have a pivot in every row:

$$\begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} \xrightarrow{R_2=R_2+hR_1} \begin{pmatrix} -1 & 0 & 2-h \\ 0 & 0 & 3+h(2-h) \end{pmatrix}.$$

The matrix has a pivot in every row unless

$$3+h(2-h) = 0, \qquad h^2-2h-3 = 0, \qquad (h-3)(h+1) = 0.$$

Therefore, T is onto as long as $h \neq 3$ and $h \neq -1$.

4. Determine which of the following transformations are linear.

a) $S : \mathbf{R}^2 \to \mathbf{R}^2$ given by $S(x_1, x_2) = (x_1,\ 3 + x_2)$

b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1 - x_2,\ x_1 x_2)$

c) $U : \mathbf{R}^2 \to \mathbf{R}^3$ given by $U(x_1, x_2) = (-x_2,\ x_1, 0)$

Solution.

a) S is not linear: $S\big((1,0)+(1,0)\big) = (2,3)$ but $S(1,0)+S(1,0) = (2,6)$.

b) T is not linear: $T(1,1)+T(1,1) = (0,2)$, but $T(2(1,1)) = T(2,2) = (0,4)$.

c) U is linear.

5. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by $60°$. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

b) Find the standard matrix for the transformation that you circled in (b).

Solution.

a) Only $U \circ T$ makes sense, as the codomain of T is $\mathbf{R}^2$, which is the domain of U.

b) The standard matrix for $U \circ T$ is

$$BA = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.$$

6. Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \\ 5 & 10 & 6 & -17 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for Col$A$.

b) Find a basis $\mathcal{B}$ for Nul$A$.

c) For each of the following vectors v, decide if v is in Nul$A$, and if so, write x as a linear combination of your basis from part (b).

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix}$$

Solution.

a) The pivot columns for A form a basis for Col$A$, so a basis is $\left\{ \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix} \right\}$.

b) We compute the parametric vector form for the general solution of $Ax = 0$:

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, a basis is given by

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

c) First we note that if

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

then $c_1 = b$ and $c_2 = d$. This makes it easy to check whether a vector is in Nul$A$.

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \neq 3 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \text{not in Nul}A. \qquad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

7. Consider $T: \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \\ x-y \end{pmatrix}$$

and $U: \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the xy-plane (forgetting the z-coordinate), then rotating counterclockwise by 90°.

a) Compute the standard matrices A and B for T and U, respectively.

b) Compute the standard matrices for $T \circ U$ and $U \circ T$.

c) Circle all that apply:

$T \circ U$ is: one-to-one onto

$U \circ T$ is: one-to-one onto

Solution.

a) We plug in the unit coordinate vectors to get

$$A = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} | & | & | \\ U(e_1) & U(e_2) & U(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

b) The standard matrix for $T \circ U$ is

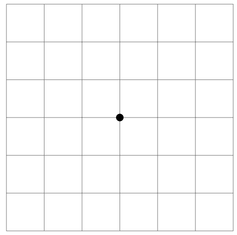
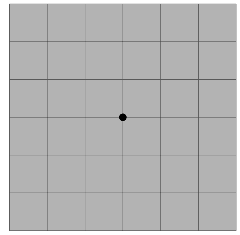
$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

The standard matrix for $U \circ T$ is

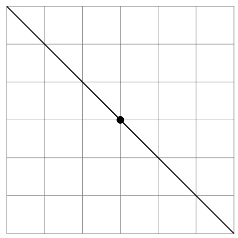
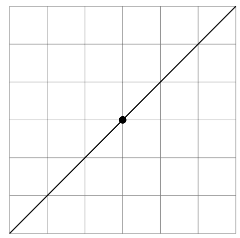
$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}.$$

c) Looking at the matrices, we see that $T \circ U$ is not one-to-one or onto, and that $U \circ T$ is one-to-one and onto.

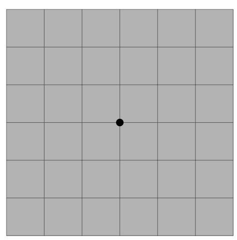
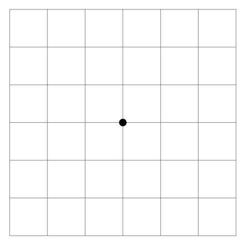
8. **a)** Write a 2×2 matrix A with **rank** 2, and draw pictures of Nul$A$ and Col$A$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Nul A =  Col A = 

b) Write a 2×2 matrix B with **rank** 1, and draw pictures of Nul$B$ and Col$B$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Nul B =  Col B = 

c) Write a 2×2 matrix C with **rank** 0, and draw pictures of Nul$C$ and Col$C$.

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Nul C =  Col C = 

(In the grids, the dot is the origin.)

9. Suppose A is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \qquad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \qquad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find A.

Solution.

The columns of A^{-1} are

$$\begin{pmatrix} A^{-1}e_1 & A^{-1}e_2 & A^{-1}e_3 \end{pmatrix} \quad \text{so} \quad A = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get A we find $(A^{-1})^{-1}$. Row-reducing $\left(A^{-1} \mid I\right)$ eventually gives us

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{so} \quad A = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$