Eigenvectors and Eigenvalues

Definition

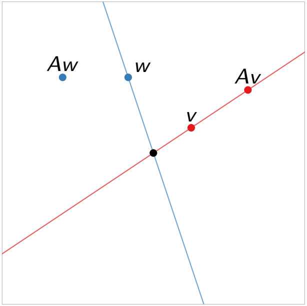
Let A be an $n \times n$ matrix.

1. An **eigenvector** of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$.
2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.
3. If λ is an eigenvalue of A, the λ-**eigenspace** is the solution set of $(A - \lambda I_n)x = 0$.

Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

- Av is a multiple of v, which means
- Av is collinear with v, which means
- Av and v are *on the same line through the origin*.

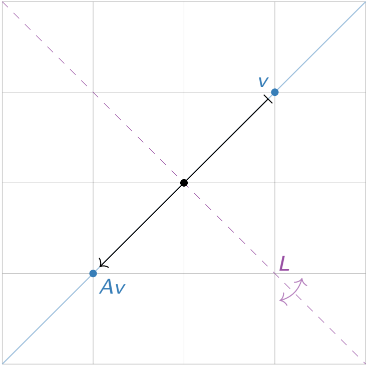


v is an eigenvector

w is not an eigenvector

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



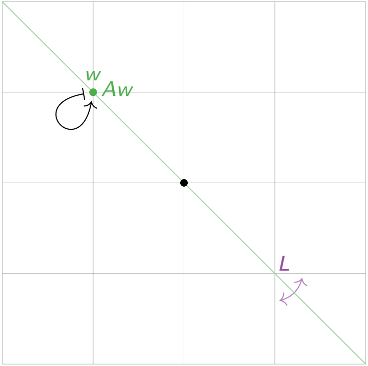
Does anyone see any eigenvectors (vectors that don't move off their line)?

v is an eigenvector with eigenvalue -1.

[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

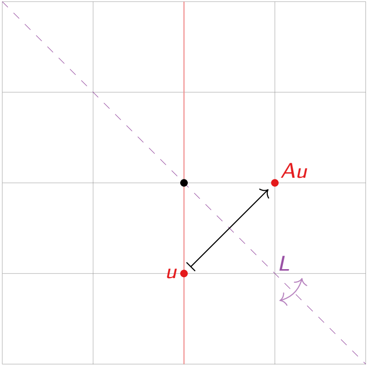
[interactive]

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

u is *not* an eigenvector.

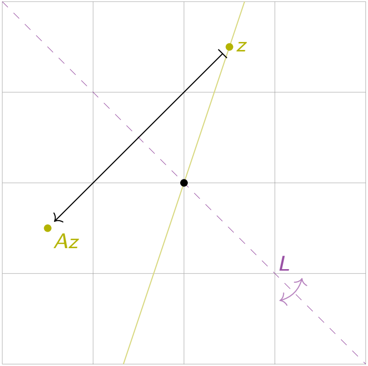
[interactive]

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

Neither is z.

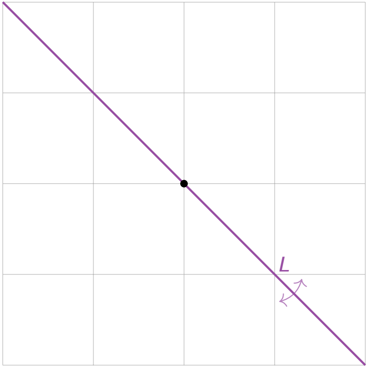
[interactive]

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



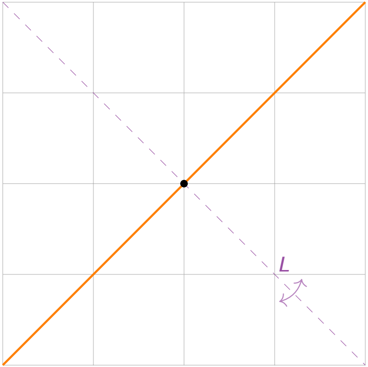
Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is L
(all the vectors x where $Ax = x$).

[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

The (-1)-eigenspace is the line $y = x$
(all the vectors x where $Ax = -x$).

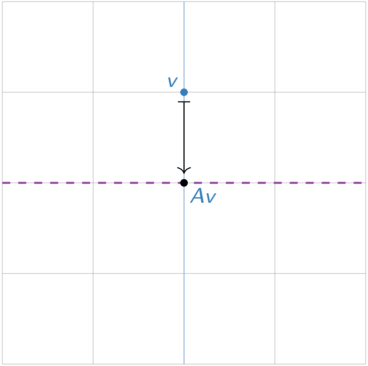
[interactive]

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

v is an eigenvector with eigenvalue 0.

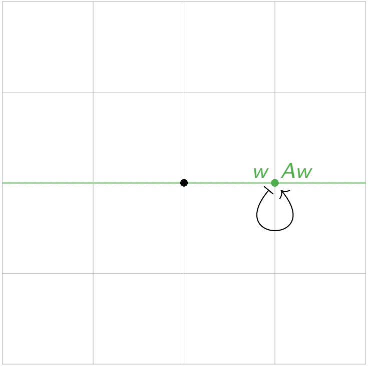
[interactive]

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



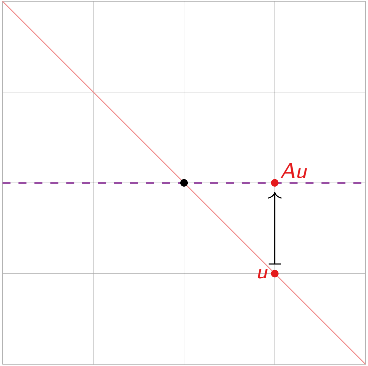
Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

u is *not* an eigenvector.

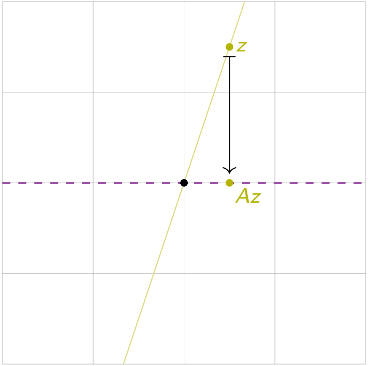
[interactive]

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



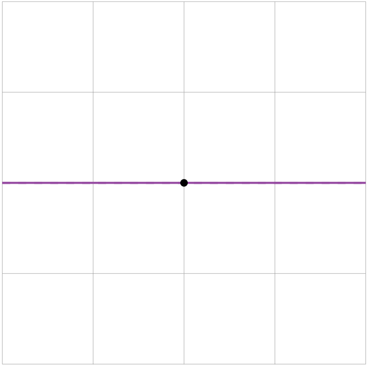
Does anyone see any eigenvectors (vectors that don't move off their line)?

Neither is z.

[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



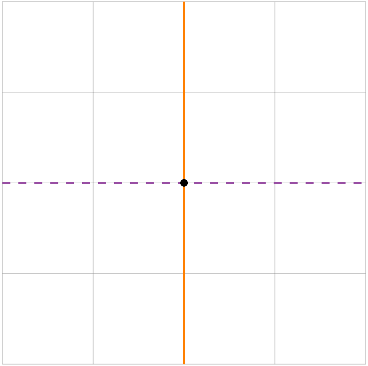
Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is the x-axis (all the vectors x where $Ax = x$).

[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

The 0-eigenspace is the y-axis
(all the vectors x where $Ax = 0x$).

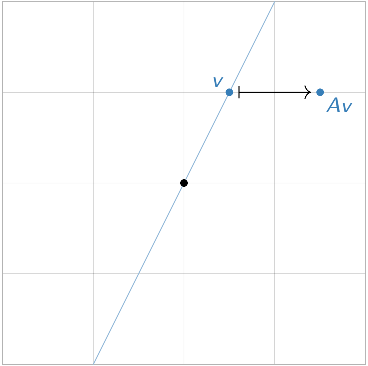
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

Vectors v above the x-axis are moved right but not up...
so they're not eigenvectors.

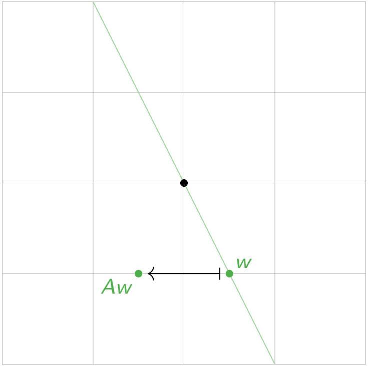
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

Vectors w below the x-axis are moved left but not down...
so they're not eigenvectors

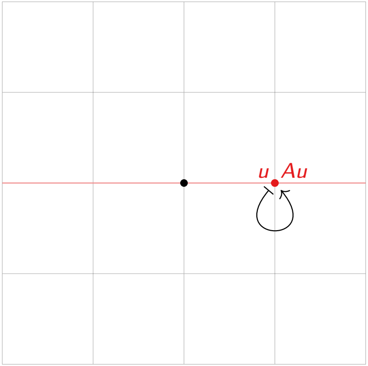
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

u is an eigenvector with eigenvalue 1.

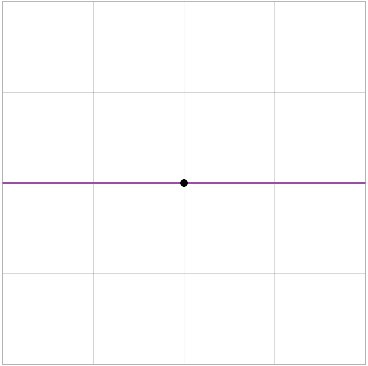
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is the x-axis (all the vectors x where $Ax = x$).

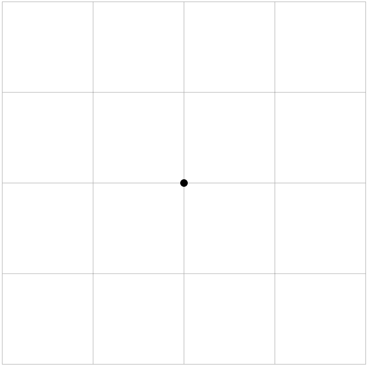
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!



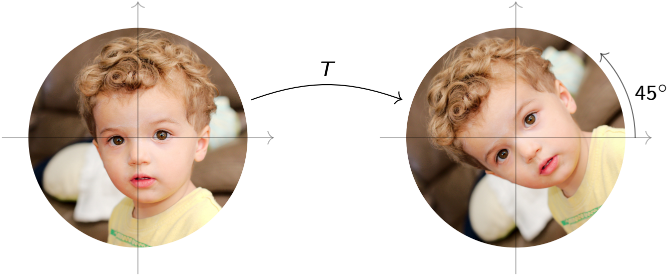
Does anyone see any eigenvectors (vectors that don't move off their line)?

There are no other eigenvectors.

[interactive]

Poll

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by 45°, and let A be the matrix for T.



Poll

Find an eigenvector of A without doing any computations.

A. Okay. **B.** No way.

Answer: **B.** No way. There are no eigenvectors of A in $\mathbf{R}^2$! [interactive]

Section 6.2

The Characteristic Polynomial

The Characteristic Polynomial

Let A be a square matrix.

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \text{ has a nontrivial solution} \\ &\iff (A - \lambda I)x = 0 \text{ has a nontrivial solution} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0. \end{aligned}$$

This gives us a way to compute the eigenvalues of A.

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det\left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \\ &= (5-\lambda)(1-\lambda) - 2\cdot 2 \\ &= \lambda^2 - 6\lambda + 1. \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}.$$

Question: What is the characteristic polynomial of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

Answer:

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

What do you notice about $f(\lambda)$?

- The constant term is $\det(A)$, which is zero if and only if $\lambda = 0$ is a root.
- The linear term $-(a+d)$ is the negative of the sum of the diagonal entries of A.

Definition

The **trace** of a square matrix A is $\mathrm{Tr}(A) =$ sum of the diagonal entries of A.

Shortcut

The characteristic polynomial of a 2×2 matrix A is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A).$$

Question: What are the eigenvalues of the rabbit population matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \\ &= 8\left(\frac{1}{4} - 0 \cdot -\lambda\right) - \lambda\left(\lambda^2 - 6 \cdot \frac{1}{2}\right) \\ &= -\lambda^3 + 3\lambda + 2. \end{aligned}$$

We know from before that one eigenvalue is $\lambda = 2$: indeed, $f(2) = -8 + 6 + 2 = 0$. Doing polynomial long division, we get:

$$\frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2.$$

Hence $\lambda = -1$ is also an eigenvalue.

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

This is not a very interesting notion *yet*. It will become interesting when we also define *geometric* multiplicity later.

Example

In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$, so the algebraic multiplicity of the eigenvalue 2 is 1, and the algebraic multiplicity of the eigenvalue -1 is 2.

Example

In the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $f(\lambda) = (\lambda - (3 - 2\sqrt{2}))(\lambda - (3 + 2\sqrt{2}))$, so the algebraic multiplicity of $3 + 2\sqrt{2}$ is 1, and the algebraic multiplicity of $3 - 2\sqrt{2}$ is 1.

Fact: If A is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree n, and its roots are the eigenvalues of A:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

Poll

True or false:
Every $n \times n$ real matrix has at least one real eigenvalue.

A. True

B. False

False. For example, if A represents rotation counterclockwise by 90° in $\mathbf{R}^2$, then A has characteristic polynomial $\lambda^2 + 1$, which has no real roots.

Factoring the Characteristic Polynomial

It's easy to factor quadraic polynomials:

$$x^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

It's less easy to factor cubics, quartics, and so on:

$$\begin{aligned} x^3 + bx^2 + cx + d = 0 &\implies x = ??? \\ x^4 + bx^3 + cx^2 + dx + e = 0 &\implies x = ??? \end{aligned}$$

Read about factoring polynomials by hand in §6.2.

Summary

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors v such that v and Av are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det\bigl(A - \lambda I\bigr)$.
- For a 2×2 matrix A, the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.