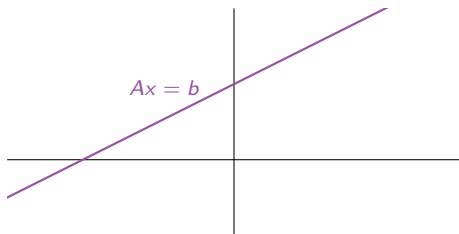


Section 3.4

Solution Sets

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.



Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Last time we discussed the set of vectors b for which $Ax = b$ has a solution.

We also described this set using spans, but it was a *different problem*.

Homogeneous Systems

Everything is easier when $b = 0$, so we start with this case.

Definition

A system of linear equations of the form $Ax = 0$ is called **homogeneous**.

These are linear equations where everything to the right of the $=$ is zero. The opposite is:

Definition

A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous**.

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution**. The nonzero solutions are called **nontrivial**.

Observation

$Ax = 0$ has a nontrivial solution

\iff there is a free variable

$\iff A$ has a column with no pivot.

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right).$$

The only solution is the trivial solution $x = 0$.

Observation

Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

This last equation is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Homogeneous Systems

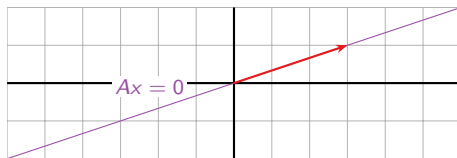
Example, continued

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in \mathbf{R} . The solution set is $\text{Span} \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



[interactive]

Note: one free variable means the solution set is a *line* in \mathbf{R}^2 ($2 = \#$ variables = $\#$ columns).

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - x_2 + 2x_3 = 0$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Homogeneous Systems

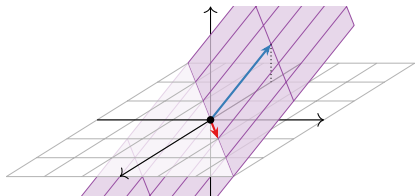
Example, continued

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

Answer: $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$



[interactive]

Note: two free variables means the solution set is a *plane* in \mathbf{R}^3 ($3 = \#$ variables = $\#$ columns).

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 - 8x_3 - 7x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Homogeneous Systems

Example, continued

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: $\text{Span} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in \mathbf{R}^4 ($4 = \#$ variables = $\#$ columns).

Parametric Vector Form

Homogeneous systems

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are x_i, x_j, x_k, \dots

Then the solutions to $Ax = 0$ can be written in the form

$$x = x_i v_i + x_j v_j + x_k v_k + \dots$$

for some vectors v_i, v_j, v_k, \dots in \mathbf{R}^n , and any scalars x_i, x_j, x_k, \dots

The solution set is

$$\text{Span}\{v_i, v_j, v_k, \dots\}.$$

The [equation](#) above is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

- A. 0
- B. 1
- C. ∞

The trivial solution is always a solution to a homogeneous system, so answer A is impossible.

This matrix has only one solution to $Ax = 0$: [\[interactive\]](#)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This matrix has infinitely many solutions to $Ax = 0$: [\[interactive\]](#)

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Inhomogeneous Systems

Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{\text{equation}} x_1 - 3x_2 = -3$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 3x_2 - 3 \\ x_2 = x_2 + 0 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

The only difference from the homogeneous case is the constant vector $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

Note that p is itself a solution: take $x_2 = 0$.

Inhomogeneous Systems

Example, continued

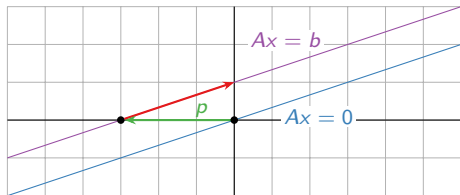
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in \mathbf{R} .

This is a *translate* of $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

[interactive]

Inhomogeneous Systems

Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

equation

$$\xrightarrow{\text{~~~~~}} x_1 - x_2 + 2x_3 = 1$$

parametric form

$$\xrightarrow{\text{~~~~~}} \begin{cases} x_1 = x_2 - 2x_3 + 1 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

parametric vector form

$$\xrightarrow{\text{~~~~~}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Inhomogeneous Systems

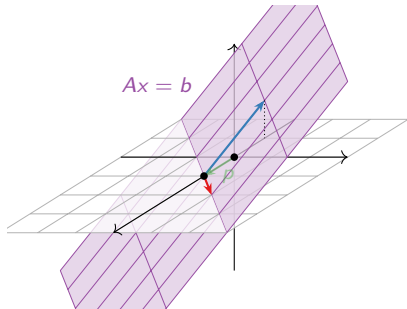
Example, continued

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

Answer: $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$



The solution set is a *translate* of

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} :$$

it is the parallel plane through

$$p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

[interactive]

Homogeneous vs. Inhomogeneous Systems

Key Observation

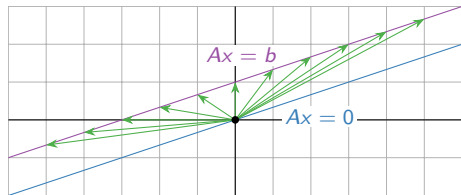
The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Why? If $Ap = b$ and $Ax = 0$, then

$$A(p + x) = Ap + Ax = b + 0 = b,$$

so $p + x$ is also a solution to $Ax = b$.

We know the solution set of $Ax = 0$ is a span. So the solution set of $Ax = b$ is a *translate* of a span: it is *parallel* to a span. (Or it is empty.)



This works for *any* specific solution p : it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

[[interactive 1](#)]

[[interactive 2](#)]

Very Important

Let A be an $m \times n$ matrix. There are now two *completely different* things you know how to describe using spans:

- ▶ The **solution set**: for fixed b , this is all x such that $Ax = b$.
 - ▶ This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
 - ▶ Lives in \mathbf{R}^n .
 - ▶ Computed by finding the parametric vector form.
- ▶ The **column span**: this is all b such that $Ax = b$ is consistent.
 - ▶ This is the span of the columns of A .
 - ▶ Lives in \mathbf{R}^m .

Don't confuse these two geometric objects!

Often, much of the first midterm tests whether you understand both.

[interactive]

Summary

- ▶ The solution set to a **homogeneous** system $Ax = 0$ is a span. It always contains the **trivial solution** $x = 0$.
- ▶ The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- ▶ The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- ▶ The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.