

Chapter 3

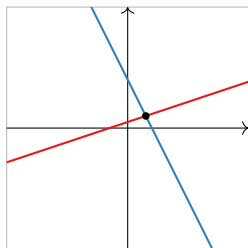
Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$\begin{aligned}x - 3y &= -3 \\ 2x + y &= 8\end{aligned}$$

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This will give us better insight into the properties of systems of equations and their solution sets.

**Remember:** I expect you to be able to draw pictures!

# Section 3.1

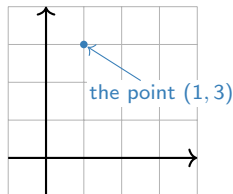
## Vectors

# Points and Vectors

We have been drawing elements of  $\mathbf{R}^n$  as points in the line, plane, space, etc. We can also draw them as arrows.

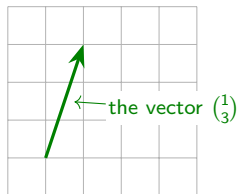
## Definition

A **point** is an element of  $\mathbf{R}^n$ , drawn as a point (a dot).



A **vector** is an element of  $\mathbf{R}^n$ , drawn as an arrow. When we think of an element of  $\mathbf{R}^n$  as a vector, we'll usually write it vertically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$



[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

However, unless otherwise specified, we'll assume a vector starts at the origin.

## Definition

- ▶ We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

- ▶ We can multiply, or **scale**, a vector by a real number  $c$ :

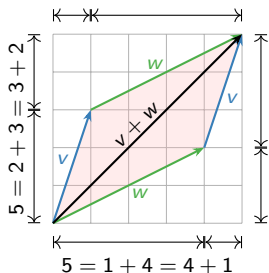
$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

We call  $c$  a **scalar** to distinguish it from a vector. If  $v$  is a vector and  $c$  is a scalar,  $cv$  is called a **scalar multiple** of  $v$ .

(And likewise for vectors of length  $n$ .) For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \quad \text{and} \quad -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$

# Vector Addition and Subtraction: Geometry



## The parallelogram law for vector addition

Geometrically, the sum of two vectors  $v, w$  is obtained as follows: place the tail of  $w$  at the head of  $v$ . Then  $v + w$  is the vector whose tail is the tail of  $v$  and whose head is the head of  $w$ . Doing this both ways creates a **parallelogram**. For example,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Why? The width of  $v + w$  is the sum of the widths, and likewise with the heights. [\[interactive\]](#)

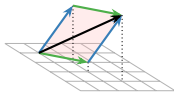
## Vector subtraction

Geometrically, the difference of two vectors  $v, w$  is obtained as follows: place the tail of  $v$  and  $w$  at the same point. Then  $v - w$  is the vector from the head of  $w$  to the head of  $v$ . For example,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Why? If you add  $v - w$  to  $w$ , you get  $v$ . [\[interactive\]](#)

This works in higher dimensions too!

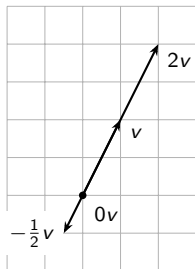


# Scalar Multiplication: Geometry

## Scalar multiples of a vector

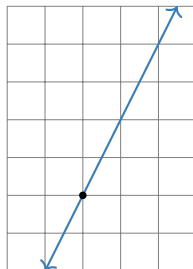
These have the same *direction* but a different *length*.

Some multiples of  $v$ .



$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$
$$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$
$$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of  $v$ .



[interactive]

So the scalar multiples of  $v$  form a *line*.



# Linear Combinations

We can add and scalar multiply in the same equation:

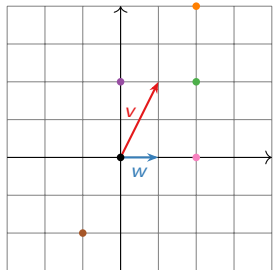
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where  $c_1, c_2, \dots, c_p$  are scalars,  $v_1, v_2, \dots, v_p$  are vectors in  $\mathbf{R}^n$ , and  $w$  is a vector in  $\mathbf{R}^n$ .

## Definition

We call  $w$  a **linear combination** of the vectors  $v_1, v_2, \dots, v_p$ . The scalars  $c_1, c_2, \dots, c_p$  are called the **weights** or **coefficients**.

## Example



Let  $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

What are some linear combinations of  $v$  and  $w$ ?

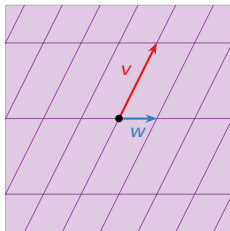
- ▶  $v + w$
- ▶  $v - w$
- ▶  $2v + 0w$
- ▶  $2w$
- ▶  $-v$

[interactive: 2 vectors]

[interactive: 3 vectors]

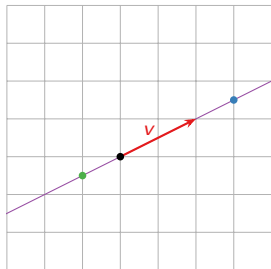
Poll

Is there any vector in  $\mathbf{R}^2$  that is *not* a linear combination of  $v$  and  $w$ ?



(The purple lines are to help measure *how much* of  $v$  and  $w$  you need to get to a given point.)

## More Examples

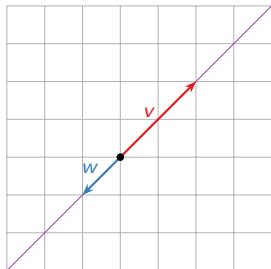


What are some linear combinations of  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ?

- ▶  $\frac{3}{2}v$
- ▶  $-\frac{1}{2}v$
- ▶ ...

What are *all* linear combinations of  $v$ ?

All vectors  $cv$  for  $c$  a real number. I.e., all *scalar multiples* of  $v$ . These form a *line*.



### Question

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

**Answer:** The line which contains both vectors.

What's different about this example and the one on the poll? [\[interactive\]](#)

## Section 3.2

### Vector Equations and Spans

# Systems of Linear Equations

## Question

Is  $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$  a linear combination of  $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$ ?

**This means:** can we solve the equation

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

where  $x$  and  $y$  are the unknowns (the coefficients)? Rewrite:

$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

This is just a system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

# Systems of Linear Equations

Continued

$$x - y = 8$$

$$2x - 2y = 16$$

$$6x - y = 3$$

matrix form  
~~~~~>

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right)$$

row reduce
~~~~~>

$$\left( \begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array} \right)$$

solution  
~~~~~>

$$x = -1$$

$$y = -9$$

Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

[interactive] ← (this is the picture of a *consistent* linear system)

What is the relationship between the original vectors and the matrix form of the linear equation? They have the same columns!

Shortcut: You can make the augmented matrix without writing down the system of linear equations first.

Summary

The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where v_1, v_2, \dots, v_p, b are vectors in \mathbf{R}^n and x_1, x_2, \dots, x_p are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{c|c|c|c|c} | & | & \cdots & | & | \\ v_1 & v_2 & & v_p & b \\ | & | & & | & | \end{array} \right),$$

where the v_i 's and b are the columns of the matrix.

So we now have (at least) *two* equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.

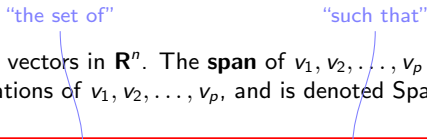
It is important to know what are *all* linear combinations of a set of vectors v_1, v_2, \dots, v_p in \mathbf{R}^n : it's exactly the collection of all b in \mathbf{R}^n such that the vector equation (in the unknowns x_1, x_2, \dots, x_p)

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = b$$

has a solution (i.e., is consistent).

Definition

Let v_1, v_2, \dots, v_p be vectors in \mathbf{R}^n . The **span** of v_1, v_2, \dots, v_p is the collection of all linear combinations of v_1, v_2, \dots, v_p , and is denoted $\text{Span}\{v_1, v_2, \dots, v_p\}$. In symbols:


$$\text{Span}\{v_1, v_2, \dots, v_p\} = \{x_1 v_1 + x_2 v_2 + \dots + x_p v_p \mid x_1, x_2, \dots, x_p \text{ in } \mathbf{R}\}.$$

Synonyms: $\text{Span}\{v_1, v_2, \dots, v_p\}$ is the subset **spanned by** or **generated by** v_1, v_2, \dots, v_p .

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of v_1, v_2, \dots, v_p .
2. The vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = b$$

has a solution.

3. The linear system with augmented matrix

$$\left(\begin{array}{c|c|c|c|c} | & | & & | & | \\ \hline v_1 & v_2 & \cdots & v_p & b \\ \hline | & | & & | & | \end{array} \right)$$

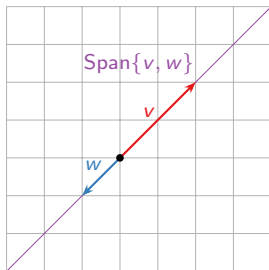
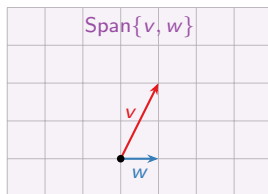
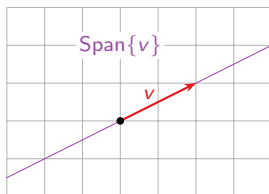
is consistent.

[[interactive example](#)] ← (this is the picture of an *inconsistent* linear system)

Note: **equivalent** means that, for any given list of vectors v_1, v_2, \dots, v_p, b , *either* all three statements are true, *or* all three statements are false.

Pictures of Span

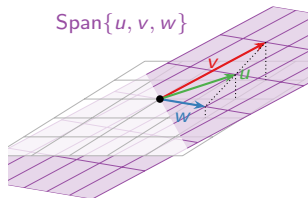
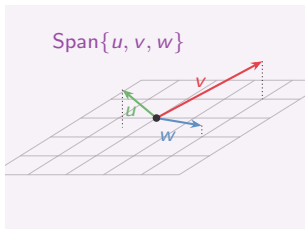
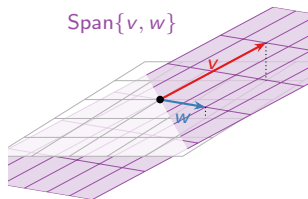
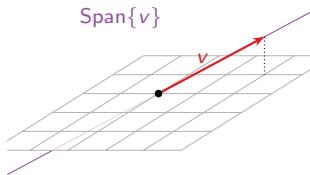
Drawing a picture of $\text{Span}\{v_1, v_2, \dots, v_p\}$ is the same as drawing a picture of all linear combinations of v_1, v_2, \dots, v_p .



[interactive: span of two vectors in \mathbb{R}^2]

Pictures of Span

In \mathbb{R}^3



[interactive: span of two vectors in \mathbb{R}^3]

[interactive: span of three vectors in \mathbb{R}^3]

How many vectors are in $\text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?

- A. Zero
- B. One
- C. Infinity

In general, it appears that $\text{Span}\{v_1, v_2, \dots, v_p\}$ is the smallest “linear space” (line, plane, etc.) containing the origin and all of the vectors v_1, v_2, \dots, v_p .

We will make this precise later.

The whole lecture was about drawing pictures of systems of linear equations.

- ▶ **Points** and **vectors** are two ways of drawing elements of \mathbf{R}^n . Vectors are drawn as arrows.
- ▶ Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- ▶ A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- ▶ The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- ▶ A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.