Math 1553 Worksheet §5.1, 5.2

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that A is an $n \times n$ matrix.

a) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

b) If v_1 and v_2 are linearly independent eigenvectors of A, then they must correspond to different eigenvalues.

Solution.

a) False. The RREF of A gives us almost no info about eigenvalues or eigenvectors. To get the eigenvectors corresponding to an eigenvalue λ, we put $A-\lambda I$ into RREF and write the solutions of $\left(A-\lambda I \mid 0\right)$ in parametric vector form.

b) False. For example, if $A = I_2$ then e_1 and e_2 are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

2. In what follows, T is a linear transformation with matrix A. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

a) $T =$ projection onto the xz-plane in $\mathbf{R}^3$.

b) $T =$ reflection over $y = 2x$ in $\mathbf{R}^2$.

Solution.

a) $T(x, y, z) = (x, 0, z)$, so T fixes every vector in the xz-plane and destroys every vector of the form $(0, a, 0)$ with a real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$. The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of x and z is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$. In other words:

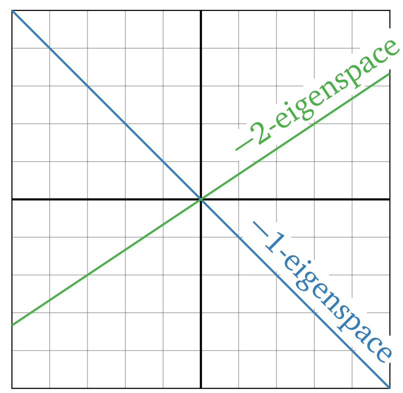
The 1-eigenspace consists of all vectors in $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, while the 0-eigenspace consists of all vectors in $\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

b) T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2,1) = (2,-1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

3. Consider the matrix
$$A = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$
Find, draw, and label the eigenspaces of A.

To save time, you may use the fact that the characteristic polynomial of A is
$$\det(A - \lambda I) = (\lambda + 2)(\lambda + 1).$$



Solution.

We were given the characteristic polynomial to save us time, but we could have computed it directly to find its roots:

$$0 = \det(A - \lambda I) = \left(-\frac{8}{5} - \lambda\right)\left(-\frac{7}{5} - \lambda\right) - \left(-\frac{2}{3}\right)\left(-\frac{3}{5}\right) = \frac{56}{25} + 3\lambda + \lambda^2 - \frac{6}{25}$$
$$= \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1), \quad \text{so the eigenvalues are} \quad \lambda = -2, \quad \lambda = -1.$$

$$\left(A + 2I \mid 0\right) = \left(\begin{array}{cc|c} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right); \text{ (−2)-eigensp. has basis } \left\{\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}\right\}.$$

$$\left(A + I \mid 0\right) = \left(\begin{array}{cc|c} -\frac{3}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & -\frac{2}{5} & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right); \text{ (−1)-eigensp. has basis } \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}.$$