Math 1553 Worksheet §§2.3-2.5

Solutions

For problems 1 and 2 below: The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{r} \\ \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array}
\begin{array}{c} \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

Feel free to use a calculator to carry out arithmetic in problems 1 and 2.

1. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3. Write a matrix equation $Ax = b$ to relate your final grades to your scores.

Solution.

In the above grading schemes, you would receive the following final grades:

$$\begin{aligned}
\text{Scheme 1:} \quad & 0.1x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 = b_1 \\
\text{Scheme 2:} \quad & 0.1x_1 + 0.1x_2 + 0.4x_3 + 0.4x_4 = b_2 \\
\text{Scheme 3:} \quad & 0.1x_1 + 0.1x_2 + 0.6x_3 + 0.2x_4 = b_3
\end{aligned}$$

This is the same as the matrix equation

$$(*) \qquad \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

2. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Use Problem 1 to determine which grading scheme leaves you with the highest overall course grade.

Solution.

According to equation (*) above, your final grades would be

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} .90 \\ .90 \\ .85 \\ .95 \end{pmatrix} = \begin{pmatrix} .89 \\ .90 \\ .88 \end{pmatrix}.$$

Hence the second grading scheme gives you the best final grade.

3. **a)** True or false. Justify your answer:
If A is a 5×4 matrix, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^5$.

TRUE **FALSE**

b) Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Must it be true that $\{v_1, v_2, v_3\}$ is linearly dependent? If so, write a linear dependence relation for the vectors.

Solution.

a) True. If A is a 5×4 matrix, then A can have at most 4 pivots (since no row or column can have more than 1 pivot). But A has 5 rows, so this means A cannot have a pivot in each row, and therefore $Ax = b$ must be inconsistent for at least one b in $\mathbf{R}^5$.

b) Yes. By definition of matrix multiplication, $-3v_1 + 2v_2 + 7v_3 = 0$, so $\{v_1, v_2, v_3\}$ is linearly dependent and the equation gives a linear dependence relation.

4. Find the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 3$ and write them in parametric vector form. How do the solution sets compare geometrically?

Solution.

The equation $x_1 - 3x_2 + 5x_3 = 0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

The solution set for $x_1 - 3x_2 + 5x_3 = 0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The equation $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.