MATH 1553, JANKOWSKI
MIDTERM 2, FALL 2019

Name		Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As usual: RREF means "reduced row echelon form."
- As usual: $e_1, e_2, \ldots, e_n$ denote the standard unit coordinate vectors in $\mathbf{R}^n$.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page. use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

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Problem 1.

[10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer, and there is no partial credit.

a) **T** **F** Suppose A is an $n \times n$ matrix and the equation $Ax = b$ has infinitely many solutions for some b in $\mathbf{R}^n$. Then A is not invertible.

b) **T** **F** If A is an $n \times n$ matrix and the transformation given by $T(x) = Ax$ is invertible, then $\mathrm{Nul}(A) = \mathrm{Col}(A)$.

c) **T** **F** The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 3y + 2z + w = 0 \right\}$ is a 3-dimensional subspace of $\mathbf{R}^4$.

d) **T** **F** Suppose A is an $m \times n$ matrix and $m < n$. Then the matrix transformation $T(x) = Ax$ is not onto.

e) **T** **F** If A is a $k \times 5$ matrix and the columns of A form a basis for $\mathbf{R}^k$, then $k = 5$.

Solution.

a) True, by the Invertible Matrix Theorem.

b) False. If A is invertible then $\mathrm{Nul}(A)$ only contains the zero vector.

c) True. It is $\mathrm{Nul}\begin{pmatrix} 1 & -3 & 2 & 1 \end{pmatrix}$ which is a subspace of $\mathbf{R}^4$ and gives 3 free variables in the corresponding homogeneous system.

d) False. It might be onto, for example when A is the 2×3 matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

e) True. Since A is $k \times 5$ we know A has 5 columns. If the 5 columns form a basis for $\mathbf{R}^k$ then they are linearly independent and span $\mathbf{R}^k$, so $\dim(\mathbf{R}^k) = 5$ hence $k = 5$.

Extra space for scratch work on problem 1

Problem 2. [12 points]

Short answer. Show your work in part (a).

a) Let $A = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix}$. Find A^{-1}.

b) Suppose A is a 4×5 matrix. Which of the following statements must be true? Clearly circle all that apply.

(i) If $\dim(\text{Nul } A) = 2$, then $\text{Col}(A) = \mathbf{R}^3$.

(ii) The matrix transformation $T(x) = Ax$ is not one-to-one.

c) Suppose $T : \mathbf{R}^4 \to \mathbf{R}^3$ is an *onto* linear transformation with standard matrix A, so $T(x) = Ax$. Which of the following statements *must* be true? Clearly circle all that apply.

(i) The matrix A has exactly three pivot columns.

(ii) For each vector v in $\mathbf{R}^3$, there is at least one vector x in $\mathbf{R}^4$ so that $T(x) = v$.

(iii) For each vector x in $\mathbf{R}^4$, there is a vector v in $\mathbf{R}^3$ so that $T(x) = v$.

(iv) $\text{Span}\left\{T(e_1),\ T(e_2),\ T(e_3),\ T(e_4)\right\} = \mathbf{R}^3$.

d) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x \geq y \right\}$. Which properties of a subspace does V satisfy? Clearly circle all that apply.

(i) V contains the zero vector.

(ii) V is closed under addition.

(iii) V is closed under scalar multiplication.

Solution to problem 2

(a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then A is invertible if $ad - bc \neq 0$, and $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$A^{-1} = \frac{1}{3-(-6)}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ -2/9 & 1/9 \end{pmatrix}.$$

It is fine if the student leaves it in either of the last two forms.

(b) (i) is NOT true: Col A is a 3-dimensional subspace of $\mathbf{R}^5$, it is NOT equal to $\mathbf{R}^3$.
(ii) is true, since A cannot have a pivot in every column (5 columns, max of 4 pivots).

(c) All 4 are true! Note that (iii) is true of any transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$. Part (iv) is just the statement that the columns of A span $\mathbf{R}^3$, which is true since T is onto.

(d) (i) is true: V contains the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since $0 \geq 0$.

(ii) is true: Suppose $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are in V. Since $x_1 \geq y_1$ and $x_2 \geq y_2$, we have

$$x_1 + x_2 \geq y_1 + y_2.$$

(iii) is not true: For example $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is in V since $5 > 1$, but $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ is not in V since $-5 < -1$.

Geometrically, V is just the region of $\mathbf{R}^2$ that lies on and beneath the line $y = x$.

Problem 3. [10 points]

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by 45°, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by

$$U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 \\ x_2 \\ x_2 - x_1 \end{pmatrix}.$$

a) Write the standard matrix A for T. Simplify your answer (do not leave it in terms of sines and cosines).

$$A = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

b) Find the standard matrix B for U. $B = \begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$.

c) Circle the composition that makes sense: $T \circ U$ $\boxed{U \circ T}$.

d) (Unrelated to (a) through (c))
Let $R : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first performs reflection across the line $y = x$, then does reflection across the y-axis. Find the standard matrix C for R.

Write G for the matrix that reflects across $y = x$, so $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Write H for the matrix that reflects across the y-axis, so $H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

We want G followed by H, so $C = HG$.

$$C = HG = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Extra space for work on problem 3

Problem 4.

[10 points]

Let $A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ -2 & 2 & -4 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

a) Find a basis for the null space of A.

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -2 & 0 \\ -2 & 2 & -4 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2=R_2+2R_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1=R_1-2R_2]{R_2\leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We see x_2 and x_4 are free, $x_3 = 0$, and $x_1 = x_2 + 2x_4$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 + 2x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for Nul A is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

b) Write a basis for the column space of A, and write dim(Col A). You do not need to show your work on this part.

The pivot columns form a basis for Col A, so a basis is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

Other answers are possible. Since a basis for Col A has 2 vectors, we know dim(Col A) $= 2$.

c) Find a vector b in $\mathbf{R}^3$ which is not in Col A. Briefly show your work.

Many answers possible. Any vector in $\mathbf{R}^3$ that does not have its second entry as (-2 times its first) will fail to be in Col$A$, for example

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 25 \\ 1 \\ 7 \end{pmatrix}.$$

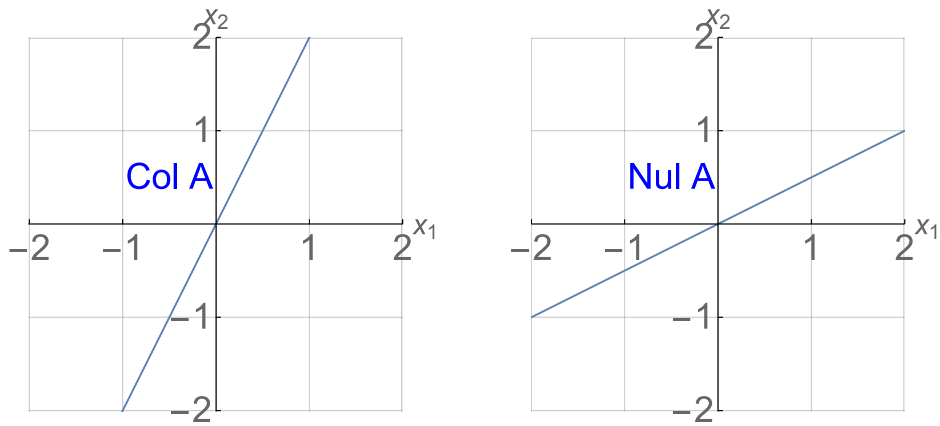
Extra space for work on problem 4

Problem 5.

[8 points]

Problems (a) and (b) are unrelated.

a) Find the matrix A so that Col A and Nul A are given below.



We need Col $A = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and Nul$A$ to be the line $x_2 = \dfrac{x_1}{2}$, so $x_1 = 2x_2$. Many answers are possible. For example,

$$A = \begin{pmatrix}1 & -2\\ 2 & -4\end{pmatrix}.$$

b) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear transformation satisfying

$$T\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

Find $T\begin{pmatrix}4\\2\\-4\end{pmatrix}$.

$\begin{pmatrix}4\\2\\-4\end{pmatrix} = 4\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix}$, so by linearity:

$$T\begin{pmatrix}4\\2\\-4\end{pmatrix} = T\left(4\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = 4T\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2T\begin{pmatrix}0\\1\\0\end{pmatrix} = 4\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\10\end{pmatrix}.$$

Extra space for work on problem 5