MATH 1553, FALL 2019
SAMPLE MIDTERM 2A: COVERS SECTIONS 2.6 THROUGH 3.6

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form.”
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §2.6 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§2.6 through 3.6.

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Problem 1.

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer.

a) **T** **F** If A is a 5×3 matrix and B is a 4×5 matrix, then the transformation $T(x) = BAx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^4$.

b) **T** **F** There is a 4×7 matrix A that satisfies $\dim(\mathrm{Nul}A) = 1$.

c) **T** **F** Suppose A is an $n \times n$ matrix and the matrix transformation T given by $T(x) = Ax$ is onto. Then T must also be one-to-one.

d) **T** **F** Suppose A is an $n \times n$ matrix and $Ax = 0$ has only the trivial solution. Then each b in $\mathbf{R}^n$ can be written as a linear combination of the columns of A.

e) **T** **F** Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^5$, so that $\mathrm{Span}\{v_1, v_2\}$ has dimension 2 and $\mathrm{Span}\{v_3, v_4\}$ has dimension 2. Then $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$ has dimension 4.

Solution.

a) True.

b) False. If $\dim(\mathrm{Nul}\ A) = 1$ then $\dim(\mathrm{Col}A) = 6$ which is absurd since $\mathrm{Col}\ A$ is a subspace of $\mathbf{R}^4$.

c) True by the Invertible Matrix Theorem.

d) True by the Invertible Matrix Theorem.

e) False. For example we could have $v_1 = v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then $\dim(\mathrm{Span}\{v_1, v_2, v_3, v_4\}) = 2$.

Extra space for scratch work on problem 1

Problem 2.

You do not need to show your work in parts (a)-(c), but show your work in (d).

a) Let $V = \text{Span}\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \right\}$.

Fill in the blank: $\dim(V) =$ ____________.

b) Suppose A is an 8×5 matrix, and the range of the transformation $T(x) = Ax$ is a line. Fill in the blank:

Nul(A) is a ________-dimensional subspace of $\mathbf{R}^{\boxed{}}$.

c) Suppose that $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix A. Which of the following conditions *guarantee* that T is one-to-one? Circle all that apply.

(i) For each x in $\mathbf{R}^n$, there is a unique y in $\mathbf{R}^m$ so that $T(x) = y$.

(ii) For each y in $\mathbf{R}^m$, the matrix equation $Ax = y$ is consistent.

(iii) The columns of A are linearly independent.

d) (3 points) Suppose A is a 2×2 matrix and $A^{-1} = \begin{pmatrix}5 & 4\\2 & 2\end{pmatrix}$. Find A.

Solution.

a) $\dim(V) = 2$ since the first two vectors span V and are linearly independent.

b) $\dim(\text{Nul } A) = 5 - \dim(\text{Col} A) = 5 - 1 = 4$, and Nul A is a subspace of $\mathbf{R}^5$.

c) Only (iii). Statement (i) is just the definition of a transformation, and (ii) defines onto.

d) $A = (A^{-1})^{-1}$, so $A = \dfrac{1}{5(2) - 4(2)} \begin{pmatrix}2 & -4\\-2 & 5\end{pmatrix} = \begin{pmatrix}1 & -2\\-1 & \frac{5}{2}\end{pmatrix}$.

Extra space for work on problem 2

Problem 3.

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ x + 2y \end{pmatrix}$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of rotation counterclockwise by $90°$.

a) Write the standard matrix A for T.

b) Write the standard matrix B for U.

c) Is T onto? YES NO

d) Is U invertible? YES NO

e) Circle the composition that makes sense: $T \circ U$ $U \circ T$

f) Write the standard matrix for the composition you chose in part (e).

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$

b) $B = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

c) Yes, T is onto since A has a pivot in each row.

d) Yes, since B has two pivots and is a 2×2 matrix.

e) $U \circ T$ makes sense.

f) $BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

Extra space for work on problem 3

Problem 4.

Parts (a) and (b) are unrelated.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ and $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$. Find the standard matrix A for T.

b) Rollo Tomasi has put the matrix A below in its reduced row echelon form:

$$A = \begin{pmatrix} 2 & 6 & -14 & 7 \\ 3 & 9 & -21 & 10 \\ 4 & 12 & -28 & 12 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 3 & -7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Write a basis for Col A (you don't need to justify your answer).

(ii) Write a new basis for Col$A$, so that no vector in your new basis is a scalar multiple of any of the vectors in the basis you wrote in part (i). Clearly show how you obtain this new basis.

(iii) Find one nonzero vector x that satisfies $Ax = 0$.

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix}$ and we're given $T(e_1) = \begin{pmatrix}1\\2\end{pmatrix}$. Using linearity of T and the fact that $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}$:

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}.$$

Thus $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$.

b) (i) The pivot columns of A are a basis for Col$A$:

$$\left\{ \begin{pmatrix}2\\3\\4\end{pmatrix}, \begin{pmatrix}7\\10\\12\end{pmatrix}. \right\}$$

In reality, the fourth column and any one among the first three columns will form a basis for Col$A$.

(ii) Many possibilities. For example, $\{w_1, w_2\}$ where $w_1 = v_1 - v_2$ and $w_2 = v_1 + v_2$.

$$w_1 = v_1 - v_2 = \begin{pmatrix}-5\\-7\\-8\end{pmatrix} \qquad w_2 = v_1 + v_2 = \begin{pmatrix}9\\13\\16\end{pmatrix}.$$

(iii) Many possibilities. It's not necessary to actually write a basis for Nul$A$ to do this, but any nonzero vector in Nul$A = \text{Span}\left\{ \begin{pmatrix}-3\\1\\0\\0\end{pmatrix}, \begin{pmatrix}7\\0\\1\\0\end{pmatrix} \right\}$ is correct.

Extra space for work on problem 4

Problem 5.

Parts (a) and (b) are unrelated. You don't need to justify your answers in part (a).

a) Consider the set $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy \geq 0 \right\}$.

(i) Does V contain the zero vector? YES NO

(ii) Is V closed under addition? YES NO

(iii) Is V closed under scalar multiplication? YES NO

b) Consider the subspace W of $\mathbf{R}^3$ given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 5y + 6z = 0 \right\}.$$

(i) Find a basis for W.

(ii) Is there a matrix A so that $\mathrm{Col}(A) = W$? If your answer is yes, write such a matrix A. If your answer is no, justify why there is no such matrix A.

Solution.

a) (i) Yes since $0(0) = 0 \geq 0$.

(ii) No. For example, $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ is in V and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is in V, but

$$\begin{pmatrix} -5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \text{ which is not in } V \text{ since } (-3)(1) < 0.$$

(iii) Yes. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in V and c is any real number then $\begin{pmatrix} cx \\ cy \end{pmatrix}$ is in V since $xy \geq 0$ and

$$(cx)(cy) = c^2 xy = c^2(\text{something 0 or greater}) \geq 0.$$

Geometrically, $xy \geq 0$ means $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the first or third quadrant. This region contains 0, is not closed under addition, and is closed under scalar multiplication.

b) (i) $W = \mathrm{Nul}\begin{pmatrix} 1 & -5 & 6 \end{pmatrix}$, so $x = 5y - 6z$ where y and z are free. A basis is

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Certainly. For example,

$$A = \begin{pmatrix} 5 & -6 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Extra space for work on problem 5