MATH 1553, FALL 2019
SAMPLE MIDTERM 2B: COVERS SECTIONS 2.6 THROUGH 3.6

Name		Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §2.6 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§2.6 through 3.6.

Problem 1.

In what follows, A is a matrix, and $T(x) = Ax$ is its matrix transformation.

Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** The zero vector is in the range of T.

b) **T** **F** If A is a non-invertible square matrix, then two of the columns of A are scalar multiples of each other.

c) **T** **F** If A is a 2×5 matrix, then $\mathrm{Nul} A$ is a subspace of $\mathbf{R}^2$.

d) **T** **F** If A has more columns than rows, then T is not onto.

e) **T** **F** If T is one-to-one and onto, then A is invertible

Solution.

a) True: $T(0) = 0$.

b) False: for instance, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is not invertible.

c) False: it is a subspace of $\mathbf{R}^5$.

d) False: however, if A has more rows than columns, then T is not onto.

e) True: the hypothesis implies that A is square, so we can apply the Invertible Matrix Theorem.

Problem 2.

Which of the following are subspaces of $\mathbf{R}^4$ and **why?**

a) $\mathrm{Span}\left\{\begin{pmatrix}1\\0\\3\\2\end{pmatrix}, \begin{pmatrix}-2\\7\\9\\13\end{pmatrix}, \begin{pmatrix}144\\0\\0\\1\end{pmatrix}\right\}$

b) $\mathrm{Nul}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

c) $\mathrm{Col}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

d) $V = \left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; xy = zw\right\}$

e) The range of a linear transformation with codomain $\mathbf{R}^4$.

Solution.

a) This is a subspace of $\mathbf{R}^4$: it is a span of three vectors in $\mathbf{R}^4$, and any span is a subspace.

b) This is not a subspace of $\mathbf{R}^4$; it is a subspace of $\mathbf{R}^3$.

c) This is a subspace of $\mathbf{R}^4$: it is the span of three vectors in $\mathbf{R}^4$.

d) This is not a subspace of $\mathbf{R}^4$: it is not closed under addition. For instance,

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \text{ are in } V\text{, but } \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \text{ is not.}$$

e) This is a subspace of $\mathbf{R}^4$: it is the column space of the associated $4\times ?$ matrix, and any column space is a subspace.

Problem 3.

Consider the matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

a) Find a basis $\{v_1, v_2\}$ for $\mathrm{Col}A$.

b) What are $\mathrm{rank}(A)$ and $\dim(\mathrm{Nul}A)$?

c) Find a basis $\{w_1, w_2\}$ for $\mathrm{Col}A$, such that w_1 is a not scalar multiple of v_1 or v_2, and likewise for w_2. Justify your answer.

Solution.

a) A basis for the column space is given by the pivot columns of A:

$$\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \end{pmatrix} \right\}.$$

b) The rank is the dimension of the column space, which is 2. The rank plus the dimension of the null space equals the number of columns, so the null space has dimension 2 as well.

c) The column space is a 2-dimensional subspace of $\mathbf{R}^2$. Thus $\mathrm{Col}A = \mathbf{R}^2$, so any basis for $\mathbf{R}^2$ works. For example, we can use the standard basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Problem 4.

Consider the vectors

$$v_1 = \begin{pmatrix}1\\2\\3\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}3\\2\\1\\0\end{pmatrix} \quad v_3 = \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$$

and the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; w = 0 \right\}.$$

a) Let e_1, e_2, e_3, e_4 be the standard unit coordinate vectors of $\mathbf{R}^4$. Justify why $\{e_1, e_2, e_3\}$ is a basis for V.

b) Justify why $\{v_1, v_2, v_3\}$ is a basis for V.

c) Let $v = \begin{pmatrix}4\\4\\4\\0\end{pmatrix}$. Write v as a linear combination of the vectors v_1, v_2, and v_3.

Solution.

a) The unit coordinate vectors are always linearly independent. They span V because

$$\begin{pmatrix}x\\y\\z\\0\end{pmatrix} = xe_1 + ye_2 + ze_3.$$

Therefore they are a basis.

b) We must check that $\{v_1, v_2, v_3\}$ is linearly independent, and that it spans V. For this, we row reduce the matrix whose columns are v_1, v_2, v_3:

$$\begin{pmatrix}1&3&1\\2&2&-1\\3&1&1\\0&0&0\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&3&1\\0&-4&-3\\0&0&4\\0&0&0\end{pmatrix}.$$

Since each column has a pivot, the vectors are linearly independent. We know from (a) that $\dim V = 3$, so by the Basis Theorem, $\mathcal{B}_2$ spans V.

c) One can solve this by row reduction, but you can eyeball this to see that $v = v_1 + v_2$.

Problem 5.

Consider the matrices

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let T and U be the associated linear transformations, respectively

$$T(x) = Ax \qquad U(x) = Bx.$$

a) Fill in the boxes:

$$T : \mathbf{R}^{\square} \longrightarrow \mathbf{R}^{\square} \qquad U : \mathbf{R}^{\square} \longrightarrow \mathbf{R}^{\square}.$$

b) Is T one-to-one?

c) Find the standard matrix for U^{-1}.

d) Find the standard matrix for $U \circ T$.

Solution.

a) You can multiply A by vectors in $\mathbf{R}^2$, and the result is a vector in $\mathbf{R}^3$. Therefore,

$$T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3.$$

Likewise, you can multiply B by vectors in $\mathbf{R}^3$, and the result is a vector in $\mathbf{R}^3$. Therefore,

$$U : \mathbf{R}^3 \longrightarrow \mathbf{R}^3.$$

b) The transformation T is one-to-one if and only if A has a pivot in each column, which it does (it is already in row echelon form).

c) The matrix for U^{-1} is B^{-1}. We compute

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -14 \\ 0 & 1 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

so the inverse is

$$B^{-1} = \begin{pmatrix} 1 & 3 & -14 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) The matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}.$$

[Scratch work]