Supplemental problems: §3.4

1. Consider $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ x - y \end{pmatrix}$$

and $U : \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the xy-plane (forgetting the z-coordinate), then rotating counterclockwise by 90°.

a) Compute the standard matrices A and B for T and U, respectively.

b) Compute the standard matrices for $T \circ U$ and $U \circ T$.

c) Circle all that apply:

$T \circ U$ is: one-to-one onto

$U \circ T$ is: one-to-one onto

Solution.

a) We plug in the unit coordinate vectors to get

$$A = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} | & | & | \\ U(e_1) & U(e_2) & U(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

b) The standard matrix for $T \circ U$ is

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

The standard matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}.$$

c) Looking at the matrices, we see that $T \circ U$ is not one-to-one or onto, and that $U \circ T$ is one-to-one and onto.

2. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by 60°. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

b) Find the standard matrix for the transformation that you circled in (b).

Solution.

a) Only $U \circ T$ makes sense, as the codomain of T is $\mathbf{R}^2$, which is the domain of U.

b) The standard matrix for $U \circ T$ is

$$BA = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.$$

3. Find all matrices B that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

Solution.

B must have two rows and two columns for the above to compute, so $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We calculate

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} a-3c & b-3d \\ -3a+5c & -3b+5d \end{pmatrix}.$$

Setting this equal to $\begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$ gives us

$$\left.\begin{aligned} a - 3c &= -3 \\ -3a + 5c &= 1 \end{aligned}\right\} \xrightarrow{\text{solve}} a = 3,\ c = 2$$

and

$$\left.\begin{aligned} b - 3d &= -11 \\ -3b + 5d &= 17 \end{aligned}\right\} \xrightarrow{\text{solve}} b = 1,\ d = 4.$$

Therefore, $B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

4. Let T and U be the (linear) transformations below:

$$T(x_1, x_2, x_3) = (x_3 - x_1,\ x_2 + 4x_3,\ x_1,\ 2x_2 + x_3) \qquad U(x_1, x_2, x_3, x_4) = (x_1 - 2x_2,\ x_1).$$

a) Which compositions makes sense (circle all that apply)? $U \circ T$ $\qquad$ $T \circ U$

b) Compute the standard matrix for T and for U.

c) Compute the standard matrix for each composition that you circled in (a).

Solution.

a) $U \circ T$ makes sense, but $T \circ U$ does not.

b) Let A be the standard matrix for T and B be the standard matrix for U.

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

c) The matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -7 \\ -1 & 0 & 1 \end{pmatrix}.$$

5. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A and B are matrices and the products AB and BA are both defined, then A and B must be square matrices with the same number of rows and columns.

b) If A, B, and C are nonzero 2×2 matrices satisfying $BA = CA$, then $B = C$.

c) Suppose A is an 4×3 matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a 3×3 matrix B which is not the zero matrix and satisfies $AB = 0$.

d) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are one-to-one linear transformations. Then $U \circ T$ is one-to-one. (What if U and T are not necessarily linear?)

Solution.

a) False. For example, if A is any 2×3 matrix and B is any 3×2 matrix, then AB and BA are both defined.

b) False. Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $BC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but $B \neq C$.

c) True. If T is not one-to-one then there is a non-zero vector v in $\mathbf{R}^3$ so that

$$Av = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The 3×3 matrix $B = \begin{pmatrix} | & | & | \\ v & v & v \\ | & | & | \end{pmatrix}$ satisfies

$$AB = \begin{pmatrix} | & | & | \\ Av & Av & Av \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

d) True. Recall that a transformation S is one-to-one if $S(x) = S(y)$ implies $x = y$ (the same outputs implies the same inputs). Suppose that $U \circ T(x) = U \circ T(y)$. Then $U(T(x)) = U(T(y))$, so since U is one-to-one, we have $T(x) = T(y)$. Since T is one-to-one, this implies $x = y$. Therefore, $U \circ T$ is one-to-one. Note that this argument does not use the assumption that U and T are linear transformations.

Alternative: We'll show that $U \circ T(x) = 0$ has only the trivial solution. Let A be the matrix for U and B be the matrix for T, and suppose x is a vector satisfying $(U \circ T)(x) = 0$. In terms of matrix multiplication, this is equivalent to $ABx = 0$. Since U is one-to-one, the only solution to $Av = 0$ is $v = 0$, so $A(Bx) = 0 \implies Bx = 0$.

Since T is one-to-one, we know that $Bx = 0 \implies x = 0$. Therefore, the equation $(U \circ T)(x) = 0$ has only the trivial solution.

6. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.

a) A 3×3 matrix P, which is not the identity matrix or the zero matrix, and satisfies $P^2 = P$.

b) A 2×2 matrix A satisfying $A^2 = I$.

c) A 2×2 matrix A satisfying $A^3 = -I$.

Solution.

a) Take P to be the natural projection onto the xy-plane in $\mathbf{R}^3$, so $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If you apply P to a vector then the result will be within the xy-plane of $\mathbf{R}^3$, so applying P a second time won't change anything, hence $P^2 = P$.

b) Take A to be matrix for reflection across the line $y = x$, so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since A swaps the x and y coordinates, repeating A will swap them back to their original positions, so $AA = I$.

c) Note that $-I$ is the matrix that rotates counterclockwise by 180°, so we need a transformation that will give you counterclockwise rotation by 180° if you do it three times. One such matrix is the rotation matrix for 60° counterclockwise,

$$A = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Another such matrix is $A = -I$.

Supplemental problems: §3.5-3.6

1. **a)** Fill in: A and B are invertible $n\times n$ matrices, then the inverse of AB is ________.

b) If the columns of an $n \times n$ matrix Z are linearly independent, is Z necessarily invertible? Justify your answer.

c) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

Solution.

a) $(AB)^{-1} = B^{-1}A^{-1}$.

b) Yes. The transformation $x \to Zx$ is one-to-one since the columns of Z are linearly independent. Thus Z has a pivot in all n columns, so Z has n pivots. Since Z also has n rows, this means that Z has a pivot in every row, so $x \to Zx$ is onto. Therefore, Z is invertible.

Alternatively, since Z is an $n \times n$ matrix whose columns are linearly independent, the Invertible Matrix Theorem says that Z is invertible.

c) Yes. Since AB is an $n \times n$ matrix and $ABx = 0$ has a unique solution, the Invertible Matrix Theorem says that AB is invertible. Note A is invertible and its inverse is $B(AB)^{-1}$, since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$

Since A is invertible, $Ax = 0$ has a unique solution by the Invertible Matrix Theorem.

2. Suppose A is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix}4\\1\\0\end{pmatrix}, \qquad A^{-1}e_2 = \begin{pmatrix}3\\2\\0\end{pmatrix}, \qquad A^{-1}e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Find A.

Solution.

The columns of A^{-1} are

$$\begin{pmatrix}A^{-1}e_1 & A^{-1}e_2 & A^{-1}e_3\end{pmatrix} \quad \text{so} \quad A = \begin{pmatrix}4 & 3 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

To get A we find $(A^{-1})^{-1}$. Row-reducing $\left(A^{-1} \mid I\right)$ eventually gives us

$$\begin{pmatrix}1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0\\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{pmatrix}, \quad \text{so} \quad A = \begin{pmatrix}\frac{2}{5} & -\frac{3}{5} & 0\\ -\frac{1}{5} & \frac{4}{5} & 0\\ 0 & 0 & 1\end{pmatrix}.$$