

## Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

# Subspaces

## Reminder

**Recall:** a subspace of  $\mathbf{R}^n$  is the same thing as a span, except we haven't computed a spanning set yet.

For example,  $\text{Col } A$  and  $\text{Nul } A$  for a matrix  $A$ .

There are lots of choices of spanning set for a given subspace.

Are some better than others?

## Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

### Definition

Let  $V$  be a subspace of  $\mathbf{R}^n$ . A **basis** of  $V$  is a set of vectors  $\{v_1, v_2, \dots, v_m\}$  in  $V$  such that:

1.  $V = \text{Span}\{v_1, v_2, \dots, v_m\}$ , and
2.  $\{v_1, v_2, \dots, v_m\}$  is linearly independent.

The number of vectors in a basis is the **dimension** of  $V$ , and is written  $\dim V$ .

Note the big  
red border here

**Why** is a basis the smallest number of vectors needed to span?

Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can't span  $V$ .

### Important

A subspace has *many different* bases, but they all have the same number of vectors.

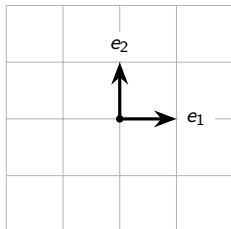
## Bases of $\mathbf{R}^2$

### Question

What is a basis for  $\mathbf{R}^2$ ?

We need two vectors that *span*  $\mathbf{R}^2$  and are *linearly independent*.  $\{e_1, e_2\}$  is one basis.

1. They span:  $\begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2$ .
2. They are linearly independent because they are not collinear.

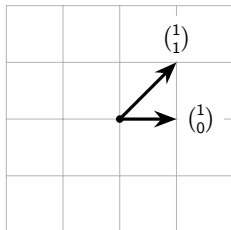


### Question

What is another basis for  $\mathbf{R}^2$ ?

Any two nonzero vectors that are not collinear.  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$  is also a basis.

1. They span:  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  has a pivot in every row.
2. They are linearly independent:  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  has a pivot in every column.



## Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for  $\mathbf{R}^n$ .  The identity matrix has columns  $e_1, e_2, \dots, e_n$ .

1. They span:  $I_n$  has a pivot in every row.
2. They are linearly independent:  $I_n$  has a pivot in every column.

**In general:**  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $\mathbf{R}^n$  if and only if the matrix

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$

has a pivot in every row and every column.

**Sanity check:** we have shown that  $\dim \mathbf{R}^n = n$ .

## Basis of a Subspace

### Example

#### Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x + 3y + z = 0 \right\} \quad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that  $\mathcal{B}$  is a basis for  $V$ . (So  $\dim V = 2$ : it is a *plane*.) [\[interactive\]](#)

0. In  $V$ : both vectors are in  $V$  because

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. **Span:** If  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  is in  $V$ , then  $y = -\frac{1}{3}(x + z)$ , so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. **Linearly independent:**

$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

## Basis for Nul $A$

Fact

The vectors in the parametric vector form of the general solution to  $Ax = 0$  always form a basis for  $\text{Nul } A$ .

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

parametric vector form  $\xrightarrow{\text{~~~~~}}$   $x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$  basis of  $\text{Nul } A$   $\xrightarrow{\text{~~~~~}}$   $\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

1. The vectors span  $\text{Nul } A$  by construction (every solution to  $Ax = 0$  has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

## Basis for Col A

### Fact

The *pivot columns* of  $A$  always form a basis for Col  $A$ .

**Warning:** I mean the pivot columns of the *original* matrix  $A$ , not the row-reduced form. (Row reduction changes the column space.)

### Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis  $\longleftrightarrow$  pivot columns in rref

So a basis for Col  $A$  is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

**Why?** See slides on linear independence.



# The Basis Theorem

## Basis Theorem

Let  $V$  be a subspace of dimension  $m$ . Then:

- ▶ Any  $m$  linearly independent vectors in  $V$  form a basis for  $V$ .
- ▶ Any  $m$  vectors that span  $V$  form a basis for  $V$ .

### Upshot

If you *already* know that  $\dim V = m$ , and you have  $m$  vectors  $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$  in  $V$ , then you only have to check *one* of

1.  $\mathcal{B}$  is linearly independent, *or*
2.  $\mathcal{B}$  spans  $V$

in order for  $\mathcal{B}$  to be a basis.

**Example:** any three linearly independent vectors form a basis for  $\mathbf{R}^3$ .

# The Rank Theorem

## Recall:

- ▶ The **dimension** of a subspace  $V$  is the number of vectors in a basis for  $V$ .
- ▶ A basis for the column space of a matrix  $A$  is given by the pivot columns.
- ▶ A basis for the null space of  $A$  is given by the vectors attached to the free variables in the parametric vector form.

## Definition

The **rank** of a matrix  $A$ , written  $\text{rank } A$ , is the dimension of the column space  $\text{Col } A$ . The **nullity** of  $A$ , written  $\text{nullity } A$ , is the dimension of the solution set of  $Ax = 0$ .

## Observe:

$\text{rank } A = \dim \text{Col } A =$  the number of columns with pivots

$\text{nullity } A = \dim \text{Nul } A =$  the number of free variables

$=$  the number of columns without pivots.

## Rank Theorem

If  $A$  is an  $m \times n$  matrix, then

$$\text{rank } A + \text{nullity } A = n = \text{the number of columns of } A.$$

In other words, [\[interactive 1\]](#) [\[interactive 2\]](#)

(dimension of column space) + (dimension of solution set) = (number of variables).

# The Rank Theorem

## Example

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & \boxed{-8} & \boxed{-7} \\ 0 & 1 & \boxed{4} & \boxed{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A free variables

A basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\},$$

so  $\text{rank } A = \dim \text{Col } A = 2$ .

Since there are two free variables  $x_3, x_4$ , the parametric vector form for the solutions to  $Ax = 0$  is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus  $\text{nullity } A = \dim \text{Nul } A = 2$ .

The Rank Theorem says  $2 + 2 = 4$ .

Poll

True or False: If  $A$  is a  $10 \times 15$  matrix and there is a basis of  $\text{Col } A$  consisting of 4 vectors, then there is a basis of  $\text{Nul } A$  consisting of 6 vectors.

**False:** if  $\text{rank } A = 4$  then  $\text{nullity } A = 15 - 4 = 11$ .

## Summary

- ▶ A **basis** of a subspace is a minimal set of spanning vectors.
- ▶ There are recipes for computing a basis for the column space and null space of a matrix.
- ▶ The **dimension** of a subspace is the number of vectors in any basis.
- ▶ The **basis theorem** says that if you already know that  $\dim V = m$ , and you have  $m$  vectors in  $V$ , then you only have to check if they span *or* they're linearly independent to know they're a basis.
- ▶ The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.