Section 2.6
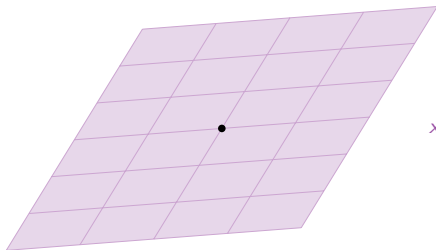
Subspaces

Today we will discuss **subspaces** of $\mathbf{R}^n$.

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.

$x + 3y + z = 0$

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V.   "not empty"
2. If u and v are in V, then $u + v$ is also in V.   "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V.   "closed under $\times$ scalars"

> **Fast-forward**
>
> Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

Definition of Subspace

Definition

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3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

What does this mean?

- If v is in V, then all scalar multiples of v are in V by (3). In other words, the line through any nonzero vector in V is also in V.
- If u, v are in V, then cu and dv are in V for any scalars c, d by (3). So $cu + dv$ is in V by (2). So $\text{Span}\{u, v\}$ is contained in V.
- Likewise, if $v_1, v_2, \ldots, v_n$ are all in V, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is contained in V: a subspace contains the span of any set of vectors in it.
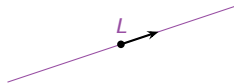
If you pick enough vectors in V, eventually their span will fill up V, so:

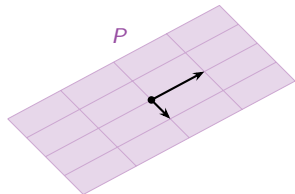> A subspace is a span of some set of vectors in it.

Examples

Example
A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.

Example
A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.

Example
All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example
The subset $\{0\}$: this subspace contains only one vector.

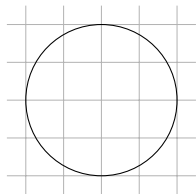Note these are all pictures of spans! (Line, plane, space, etc.)

A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.

All of the following non-examples on the next slide are still subsets.

A **subspace** is a special kind of subset, that satisfies the three defining properties.
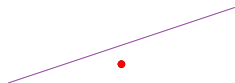

Subset: *yes*
Subspace: *no*
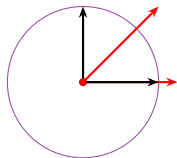
Non-Examples

Non-Example

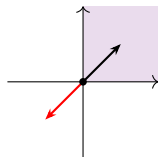A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.

Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."
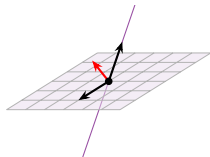
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.

Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.

Subspaces are Spans, and Spans are Subspaces

Theorem
Any $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace.

> **!!!**
>
> Every subspace is a span, and every span is a subspace.

Definition
If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_p$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.
3. Similarly, if u is in the span, then so is cu for any scalar c.

Poll

Which of the following are subspaces?
  A. The empty set $\{\}$.
  B. The solution set to a homogeneous system of linear equations.
  C. The solution set to an inhomogeneous system of linear equations.
  D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.
For the ones which are not subspaces, which property(ies) do they not satisfy?

A. This is not a subspace: it does not contain the zero vector.
B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.
C. This is not a subspace: it does not contain 0.
D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by π).

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies ab = 0$ ✔

3. Is V closed under scalar multiplication?
   - Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be (an unknown vector) in V.
   - *This means:* a and b are numbers such that $ab = 0$.
   - Let c be a scalar. Is $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$ in V?
   - *This means:* $(ca)(cb) = 0$.
   - Well, $(ca)(cb) = c^2(ab) = c^2(0) = 0$ ✔

2. Is V closed under addition?
   - Let $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} a' \\ b' \end{pmatrix}$ be (unknown vectors) in V.
   - *This means:* $ab = 0$ and $a'b' = 0$.
   - Is $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$ in V?
   - *This means:* $(a + a')(b + b') = 0$.
   - This is not true for all such a, a', b, b': for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in V, but their sum $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in V, because $1 \cdot 1 \neq 0$. ✘
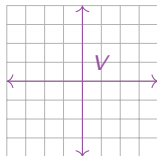
We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written Col A.

- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:
$$\text{Nul } A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$
This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace.

Check that the null space is a subspace:

1. 0 is in Nul A because $A0 = 0$.

2. If u and v are in Nul A, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
so $u + v$ is in Nul A.

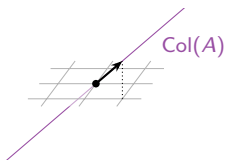3. If u is in Nul A, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in Nul A.

Column Space and Null Space
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:

$$\text{Col } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

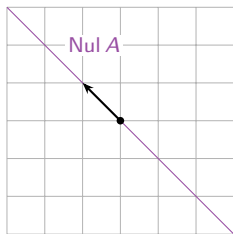This is a line in $\mathbf{R}^3$.

Col(A)

Let's compute the null space:

The reduced row echelon form of A is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$\begin{matrix} x = -y \\ y = y \end{matrix}$ $\xrightarrow{\text{parametric vector form}}$ $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

Nul A

Hence the null space is $\text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$, a line in $\mathbf{R}^2$.

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors that span the null space?

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad \begin{array}{c} \text{if, say, } x_3 \text{ and } x_4 \\ \text{are the free} \\ \text{variables. So} \end{array} \quad \text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §3.4 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

  Col A = the span of the columns of A

  Nul A = the solution set of $Ax = 0$.

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?