Math 1553 Worksheet, Chapter 7

1. True or false (justify your answer!): If u, v, w are vectors in $\mathbf{R}^n$ with $u \perp v$ and $v \perp w$, then $u \perp w$.

Solution.

False. For example, take $u = \begin{pmatrix}1\\0\end{pmatrix}$, $v = \begin{pmatrix}0\\1\end{pmatrix}$, and $w = \begin{pmatrix}2\\0\end{pmatrix}$. Then $u \perp v$ and $v \perp w$ but $u \cdot w = 2$.

2. Let W be the set of all vectors in $\mathbf{R}^3$ of the form $(x, x-y, y)$ where x and y are real numbers.

a) Find a basis for $W^\perp$.

b) Find the matrix B for orthogonal projection onto W.

c) Diagonalize B by finding an invertible matrix C and diagonal matrix D so that $B = CDC^{-1}$.

Solution.

a) A vector in W has the form

$$\begin{pmatrix}x\\x-y\\y\end{pmatrix} = \begin{pmatrix}x\\x\\0\end{pmatrix} + \begin{pmatrix}0\\-y\\y\end{pmatrix} = x\begin{pmatrix}1\\1\\0\end{pmatrix} + y\begin{pmatrix}0\\-1\\1\end{pmatrix}, \qquad \text{so } W \text{ has basis } \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}.$$

To get $W^\perp$ we find $\mathrm{Nul}\begin{pmatrix}1 & 1 & 0\\0 & -1 & 1\end{pmatrix}$ which gives us $x_1 = -x_3$, $x_2 = x_3$, and $x_3 = x_3$ (free), so $W^\perp$ has basis $\left\{\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$.

b) Let A be the matrix whose columns are the basis vectors for W: $A = \begin{pmatrix}1 & 0\\1 & -1\\0 & 1\end{pmatrix}$.

We calculate $A^TA = \begin{pmatrix}1 & 1 & 0\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix}$, so

$$B = A(A^TA)^{-1}A^T = \begin{pmatrix}1 & 0\\1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix}^{-1}\begin{pmatrix}1 & 1 & 0\\0 & -1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\1 & -1\\0 & 1\end{pmatrix}\frac{1}{3}\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & -1 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2 & 1 & 1\\1 & 2 & -1\\1 & -1 & 2\end{pmatrix}.$$

c) The basis for W is a basis for the 1-eigenspace of B, while the basis for $W^{\perp}$ is a basis for the 0-eigenspace of B. Thus $B = CDC^{-1}$ where

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Find, and draw, the best fit line $y = Mx + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= M(0) + B \\ 8 &= M(1) + B \\ 8 &= M(3) + B \\ 20 &= M(4) + B \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} M \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

Note the order of M (the slope) and B (the constant term) that we chose when forming the columns of our matrix A. This means that our least-squares answer will have first entry equal to the slope and second entry equal to the constant term of the best-fit line. We solve $A^T A\widehat{x} = A^T b$ for $\widehat{x}$.

$$A^T A = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array} \right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array} \right).$$

The least squares solution is $M = 4$ and $B = 1$, so the best fit line is $\boxed{y = 4x + 1}$.

Aside: Not all least-squares applications involve best-fit lines. Had we wanted a quadratic function to fit our data, we could have instead found the best-fit parabola $Ax^2 + Bx + C$. We would have gotten:

$$\begin{aligned} 0 &= A(0^2) + B(0) + C \\ 8 &= A(1^2) + B(1) + C \\ 8 &= A(3^2) + B(3) + C \\ 20 &= A(4^2) + B(4) + C \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

Painful computations would show that the least-squares solution is $A = 2/3$, $B = 4/3$, and $C = 2$, so the best fit quadratic is $y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$.

Below is a picture with the best-fit line and best-fit parabola. The "best fit cubic" would be the cubic $y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$, which actually passes through all four data points.