MATH 1553, FALL 2018
SAMPLE MIDTERM 3: 4.5 THROUGH 6.5

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work unless instructed otherwise. A correct answer without appropriate work will receive little or no credit. If you cannot fit your work on the front side of the page, use the back side of the page as indicated.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §4.5 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§4.5 through 6.5.

Problem 1.

[2 points each]

Answer true if the statement is *always* true. Otherwise, answer false. In every case, assume that the entries of the matrix A are real numbers.

a) **T** **F** If A is the 3×3 matrix satisfying $Ae_1 = e_2$, $Ae_2 = e_3$, and $Ae_3 = e_1$, then $\det(A) = 1$.

b) **T** **F** If A is an $n \times n$ matrix and $\det(A) = 2$, then 2 is an eigenvalue of A.

c) **T** **F** If A is an $n \times n$ matrix and its rows span $\mathbf{R}^n$, then A is invertible.

d) **T** **F** If A is an $n \times n$ matrix and v and w are eigenvectors of A, then $v + w$ is also an eigenvector of A.

e) **T** **F** If A and B are $n \times n$ matrices with $\det(A) = 0$ and $\det(B) = 0$, then $\det(A+B) = 0$.

Solution.

a) True. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. You can compute $\det(A) = 1$ or just do two row swaps to get the identity matrix, so that $\det(A) = (-1)^2 = 1$.

b) False. For example, $A = \begin{pmatrix} 4 & 0 \\ 0 & 1/2 \end{pmatrix}$ has $\det(A) = 2$ but its eigenvalues are 4 and $\frac{1}{2}$.

c) True. The rows of A are the columns of A^T and they span $\mathbf{R}^n$, so A^T is invertible by the Invertible Matrix Theorem and thus A is invertible.

d) False. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors, but $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector.

e) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Extra space for scratch work on problem 1

Problem 2. [10 points]

Short answer. Show your work on part (c). In every case, the entries of each matrix must be real numbers.

a) Write a 2×2 matrix A which is invertible but not diagonalizable.

b) Write a 2×2 matrix A for which $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors corresponding to the same eigenvalue.

c) Find the area of the triangle with vertices $(0,0)$, $(1,4)$, and $(4,2)$.

d) Write a 3×3 matrix A with only one real eigenvalue $\lambda = 4$, such that the 4-eigenspace for A is a two-dimensional plane in $\mathbf{R}^3$.

e) Suppose A is an $n \times n$ matrix. Which of the following must be true? Circle all that apply.

I. If $\det(A) = 0$ then A is not invertible.

II. If A is diagonalizable, then A has n distinct eigenvalues.

Solution.

a) Many answers possible. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

b) Any scalar multiple of the identity will work, for example $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

c) The area is $\frac{1}{2}\left|\det\begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}\right| = \frac{1}{2}|2-16| = 7$.

d) Many examples possible. For example, $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

e) (I) is correct.

Extra space for work on problem 2

Problem 3. [11 points]

Parts (a) and (b) are unrelated.

a) Consider the matrix

$$A = \begin{pmatrix} 3 & -7 \\ 1 & -1 \end{pmatrix}$$

Find all eigenvalues of A. Simplify your answer. For the eigenvalue with negative imaginary part, find an eigenvector.

b) Suppose B is a 3×3 matrix and $Be_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $Be_2 = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$, and $Be_3 = \begin{pmatrix} -4 \\ 1 \\ -5 \end{pmatrix}$.

Find B^{-1}.

Solution.

a) We compute the characteristic equation:

$$0 = \det(A - \lambda I) = (3-\lambda)(-1-\lambda) + 7 = \lambda^2 - 2\lambda - 3 + 7 = \lambda^2 - 2\lambda + 4.$$

By the quadratic formula,

$$\lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm \sqrt{3}i.$$

Let $\lambda = 1 - \sqrt{3}i$. Then

$$\left(A - \lambda I \mid 0\right) = \left(\begin{array}{cc|c} 2+\sqrt{3}i & -7 & 0 \\ \star & \star & 0 \end{array}\right) \xrightarrow[R_1 = R_1/(2+\sqrt{3}i)]{\text{destroy } R_2} \left(\begin{array}{cc|c} 1 & \frac{-7}{2+\sqrt{3}i} & 0 \\ 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -2+\sqrt{3}i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So $x_1 = (2 - \sqrt{3}i)x_2$ and x_2 is free. An eigenvector is $v = \begin{pmatrix} 2-\sqrt{3}i \\ 1 \end{pmatrix}$.

An alternative method for finding an eigenvector, using a trick you may have seen in class, is to take the first row $(a\ b)$ of $A - \lambda I_2$ to get an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$:

$$A - \lambda I_2 = \begin{pmatrix} 2+\sqrt{3}i & -7 \\ \star & \star \end{pmatrix}$$

Thus $\begin{pmatrix} 7 \\ 2+\sqrt{3}i \end{pmatrix}$ is an eigenvector for λ. This answer is equivalent to our answer v above since it is a nonzero scalar multiple of v, as $\begin{pmatrix} 7 \\ 2+\sqrt{3}i \end{pmatrix} = (2+\sqrt{3}i)v$.

b) $B = \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & 1 \\ 1 & -4 & -5 \end{pmatrix}$. Work shows that

$$\begin{pmatrix} B \mid I \end{pmatrix} = \begin{pmatrix} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -4 & -5 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & -1 & 6 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{pmatrix}$$

so $B^{-1} = \begin{pmatrix} -1 & 6 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$.

Extra space for work on problem 3

Problem 4. [9 points]

Let $A = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) Find a basis for each eigenspace of A. Mark your answers clearly.

c) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} -1-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 4-\lambda \end{pmatrix} = (2-\lambda)(-1)^4 \det\begin{pmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{pmatrix}$$

$$= (2-\lambda)((-1-\lambda)(4-\lambda)+6) = (2-\lambda)(\lambda^2 - 3\lambda - 4 + 6)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) = (2-\lambda)(\lambda - 2)(\lambda - 1)$$

So $\lambda = 1$ and $\lambda = 2$ are the eigenvalues.

$\underline{\lambda = 1}$: $\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/2]{R_3 = R_3 + \frac{3}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ with solution $x_1 = -x_3$, $x_2 = 0$, $x_3 = x_3$. The 1-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\underline{\lambda = 2}$:

$$\left(A - 2I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/3]{R_3 = R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

with solution $x_1 = -\frac{2}{3}x_3$, $x_2 = x_2$, $x_3 = x_3$. The 2-eigenspace has basis $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

b) A is diagonalizable; $A = CDC^{-1}$ where $C = \begin{pmatrix} -1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

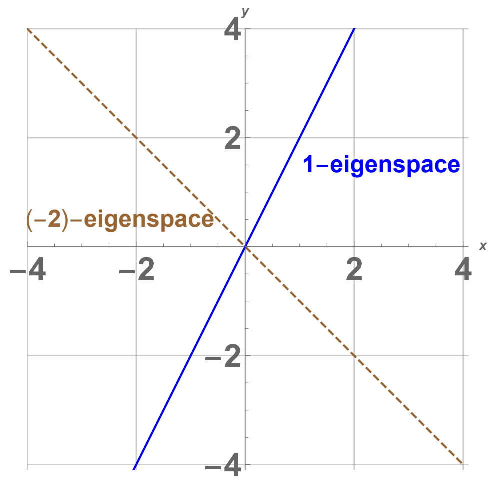
Extra space for work on problem 4

Problem 5. [10 points]

Parts (a) and (b) are not related.

a) Find $\det(A^3)$ if $A = \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 0 & -1 & 20 \end{pmatrix}$.

b) Find the 2×2 matrix A whose eigenspaces are drawn below. Fully simplify your answer. (to be clear: the dashed line is the (−2)-eigenspace).



Solution.

a) Using the cofactor expansion along the second row, we find

$$\det(A) = -2(-1)^5 \det\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & 20 \end{pmatrix} = 2\big(20 + 3(-6) + 2(-2)\big) = 2(20 - 18 - 4) = -4,$$

so $\det(A^3) = (-4)^3 = -64$.

b) From the picture, we see $\lambda_1 = 1$ has eigenvector $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Also, $\lambda = -2$ has eigenvector $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Forming $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ we get $A = CDC^{-1}$, so

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -3 & 3 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$

Extra space for work on problem 5