Math 1553, Extra Practice for Midterm 1 (through §3.4)

Solutions

1. In this problem, A is an $m \times n$ matrix (m rows and n columns) and b is a vector in $\mathbf{R}^m$. Circle **T** if the statement is always true (for any choices of A and b) and circle **F** otherwise. Do not assume anything else about A or b except what is stated.

a) **T** **F** The matrix below is in reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

b) **T** **F** If A has fewer than n pivots, then $Ax = b$ has infinitely many solutions.

c) **T** **F** If the columns of A span $\mathbf{R}^m$, then $Ax = b$ must be consistent.

d) **T** **F** If $Ax = b$ is consistent, then the solution set is a span.

Solution.

a) True.

b) False: For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has one pivot but has no solutions.

c) True: the span of the columns of A is exactly the set of all v for which $Ax = v$ is consistent. Since the span is $\mathbf{R}^m$, the matrix equation is consistent no matter what b is.

d) False: it is a *translate* of a span (unless $b = 0$).

2. **a)** Is $\begin{pmatrix}0\\1\\0\end{pmatrix}$ in the span of $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\3\\1\end{pmatrix}$? Justify your answer.

b) What best describes Span $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$? Justify your answer.

(I) It is a plane through the origin.

(II) It is three lines through the origin.

(III) It is all of $\mathbf{R}^3$.

(IV) It is a plane, plus the line through the origin and the vector $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

c) Does Span $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\3\end{pmatrix}\right\} = \mathbf{R}^3$? If yes, justify your answer. If not, write a vector in $\mathbf{R}^3$ which is not in Span $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\3\end{pmatrix}\right\}$.

Solution.

a) No. We row-reduce the corresponding augmented matrix to get

$$\left(\begin{array}{cc|c}0 & 2 & 0\\1 & 3 & 1\\1 & 1 & 0\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right)$$

which is inconsistent since it has a pivot in the right column.

b) It is all of $\mathbf{R}^3$. From the RREF in part (a), we know that the matrix $\begin{pmatrix}0 & 2 & 0\\1 & 3 & 1\\1 & 1 & 0\end{pmatrix}$ has a pivot in every row, so its columns span $\mathbf{R}^3$.

c) No. The first and third vectors are scalar multiples of each other, so we can see the three vectors cannot span $\mathbf{R}^3$. Note that any vector in the span has first coordinate equal to the negative of the third coordinate, so (for example) $\begin{pmatrix}1\\0\\2\end{pmatrix}$ is not in the span.

3. Let $v_1 = \begin{pmatrix} 1 \\ k \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ h \end{pmatrix}$.

a) Find all values of h and k so that $x_1 v_1 + x_2 v_2 = b$ has infinitely many solutions.

b) Find all values of h and k so that b is *not* in Span$\{v_1, v_2\}$.

c) Find all values of h and k so that there is exactly one way to express b as a linear combination of v_1 and v_2.

Solution.

Each part uses the row-reduction

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ k & 4 & h \end{array}\right) \xrightarrow{R_2 = R_2 - kR_1} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 4+k & h-k \end{array}\right).$$

a) The system $\left(\begin{array}{cc|c} v_1 & v_2 & b \end{array}\right)$ has infinitely many solutions if and only if the right column is not a pivot column and there is at least one free variable. This means that $4+k=0$ and $h-k=0$, so $k=-4$ and $h=k$, thus $\boxed{k=-4 \text{ and } h=-4}$.

b) The right column is a pivot column when $4+k=0$ and $h-k \neq 0$. Thus $\boxed{k=-4 \text{ and } h \neq -4}$.

c) The system will have a unique solution when the right column is not a pivot column but both other columns are pivot columns. This is when $4+k \neq 0$, so $\boxed{k \neq -4 \text{ and } h \text{ is any real number}}$.

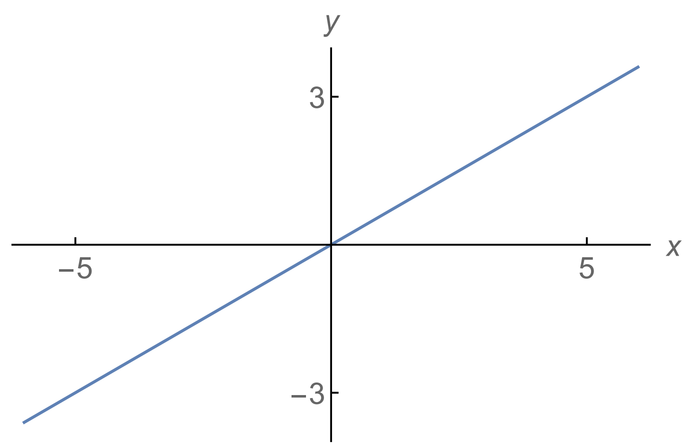
4. Let $A = \begin{pmatrix} 5 & -5 & 10 \\ 3 & -3 & 6 \end{pmatrix}$. Draw the column span of A.

Solution.

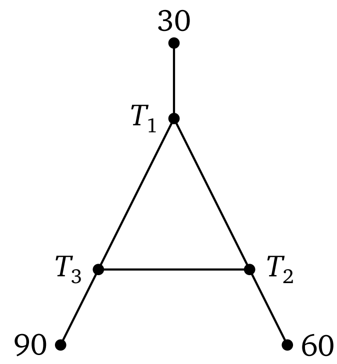
Let $v_1,\ v_2,\ v_3$ be the columns of A. The columns are scalar multiples of each other: $v_2 = -v_1$ and $v_3 = 2v_1$. This means that all three vectors are on the same line through the origin, so

$$\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1\} = \mathrm{Span}\left\{\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right\}.$$

This is the line through the origin and $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, namely the line $y = \dfrac{3x}{5}$.



5. The diagram below represents the temperature at points along wires, in celcius.



Let $T_1,\ T_2,\ T_3$ be the temperatures at the interior points. Assume the temperature at each interior point is the average of the temperatures of the three adjacent points.

a) Write a system of three linear equations whose solution would give the temperatures T_1, T_2, and T_3. Do not solve it.

b) Write the system as a vector equation. Do not solve it.

c) Write a matrix equation $Ax = b$ that represents this system. Specify every entry of A, x, and b. Do not solve it.

Solution.

a) The left side system below or right-side system below are both fine.

$$T_1 = \frac{T_2 + T_3 + 30}{3}, \quad \text{or} \quad 3T_1 - T_2 - T_3 = 30.$$
$$T_2 = \frac{T_1 + T_3 + 60}{3}, \quad \text{or} \quad -T_1 + 3T_2 - T_3 = 60.$$
$$T_3 = \frac{T_1 + T_2 + 90}{3}, \quad \text{or} \quad -T_1 - T_2 + 3T_3 = 90.$$

b) $T_1\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} + T_2\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} + T_3\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix}.$

c) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix}.$

6. For each of the following, give an example if it is possible. If it is not possible, justify why there is no such example.

a) A 3×4 matrix A in RREF with 2 pivot columns, so that for some vector b, the system $Ax = b$ has exactly three free variables.

b) A homogeneous linear system with no solution.

c) A 5×3 matrix in RREF such that $Ax = 0$ has a non-trivial solution.

Solution.

a) Not possible. If A had 2 pivot columns and 3 free variables then it would have 5 columns.

b) Not possible. Any homogeneous linear system has the trivial solution.

c) Yes. For the matrix A below, the system $Ax = 0$ will have two free variables and thus infinitely many solutions.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

7. Write an augmented matrix corresponding to a system of two linear equations in three variables x_1, x_2, x_3, whose solution set is the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$.

Briefly justify your answer.

Solution.

This problem is familiar territory, except that here, we are asked to come up with a system with the prescribed span, rather than being handed a system and discovering the span.

Since the span of any vector includes the origin, the zero vector is a solution, so the system is homogeneous.

Note that the span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ is all vectors of the form $t\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ where t is real.

It consists of all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ so that $x_1 = -4x_2,\ \ x_2 = x_2,\ \ x_3 = 0$.

The equation $x_1 = -4x_2$ gives $x_1 + 4x_2 = 0$, so one line in the matrix can be $\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \end{array}\right)$.

The equation $x_3 = 0$ translates to $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \end{array}\right)$. Note that this leaves x_2 free, as desired.

This gives us the augmented matrix

$$\boxed{\left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)}.$$

(Multiple examples are possible)

8. Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

Solution.

a) Let w, g, and d be the number of widgets, gizmos, and doodads produced.

$$\begin{pmatrix} w \\ g \\ d \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

b) We need to solve the vector equation

$$\begin{pmatrix} 16 \\ 5 \\ 3 \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

We put it into an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 10 & 4 & 16 \\ 3 & 1 & 5 \\ 2 & 1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 10 & 4 & 16 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

These equations are consistent, but they tell us that factory B would have to run for -1 hours! Therefore it can't be done.

9. Consider the system below, where h and k are real numbers.

$$\begin{aligned} x + 3y &= 2 \\ 3x - hy &= k. \end{aligned}$$

a) Find the values of h and k which make the system inconsistent.

b) Find the values of h and k which give the system a unique solution.

c) Find the values of h and k which give the system infinitely many solutions.

Solution.

We form an augmented matrix and row-reduce.

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & -h & k \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -h-9 & k-6 \end{array}\right)$$

a) The system is inconsistent precisely when the augmented matrix has a pivot in the rightmost column. This is when $-h-9=0$ and $k-6 \neq 0$, so $\boxed{h = -9 \text{ and } k \neq 6}$.

b) The system has a unique solution if and only if the left two columns are pivot columns. We know the first column has a pivot, and the second column has a pivot precisely when $-h-9 \neq 0$, so $\boxed{h \neq -9 \text{ and } k \text{ can be any real number}}$.

c) The system has infinitely many solutions when the system is consistent and has a free variable (which in this case must be y), so $-h-9=0$ and $k-6=0$, hence $\boxed{h = -9 \text{ and } k = 6}$.

10. Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) Find the parametric vector form for the general solution.

b) Find the parametric vector form of the corresponding *homogeneous* equations. [Hint: you've already done the work.]

Solution.

a) We put the equations into an augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 3 & 4 & 5 & 6 & -2 \\ 5 & 6 & 7 & 8 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & -2 & -4 & -6 & 4 \\ 0 & -4 & -8 & -12 & 8 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This means x_3 and x_4 are free, and the general solution is

$$\begin{cases} x_1 \qquad - x_3 - 2x_4 = 2 \\ \qquad x_2 + 2x_3 + 3x_4 = -2 \end{cases} \implies \begin{cases} x_1 = x_3 + 2x_4 + 2 \\ x_2 = -2x_3 - 3x_4 - 2 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

This gives the parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

b) Part (a) shows that the solution set of the original equations is the translate of

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{by} \quad \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

We know that the solution set of the homogeneous equations is the parallel plane through the origin, so it is

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Hence the parametric vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$