Math 1553 Supplement §6.1, 6.2

Supplemental Problems

1. Match the statements (i)-(v) with the corresponding statements (a)-(e). All matrices are 3×3. There is a unique correspondence. Justify the correspondences in words.

(i) $Ax = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ has a unique solution.
(ii) The transformation $T(v) = Av$ fixes a nonzero vector.
(iii) A is obtained from B by subtracting the third row of B from the first row of B.
(iv) The columns of A and B are the same; except that the first, second and third columns of A are respectively the first, third, and second columns of B.
(v) The columns of A, when added, give the zero vector.

(a) 0 is an eigenvalue of A.
(b) A is invertible.
(c) $\det(A) = \det(B)$
(d) $\det(A) = -\det(B)$
(e) 1 is an eigenvalue of A.

Solution.

(i) matches with (b).
(ii) matches with (e).
(iii) matches with (c).
(iv) matches with (d).
(v) matches with (a).

2. Find a basis $\mathcal{B}$ for the (-1)-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\left(Z - \lambda I \mid 0\right) = \left(Z + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -3+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

3. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

4. Give an example of matrices A and B which satisfy the following:

(I) A and B have the same eigenvalues, and the same algebraic multiplicities for each eigenvalue.

(II) For some eigenvalue λ, the λ-eigenspace for A has a different dimension than the λ-eigenspace for B.

Justify your answer.

Solution.

Many examples possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Both A and B have characteristic equation $\lambda^2 = 0$, so each has eigenvalue $\lambda = 0$ with algebraic multiplicity two. However, the 0-eigenspace for A is $\mathbf{R}^2$ and thus has dimension 2, while the 0-eigenspace for B has dimension 1 (the line $y = 0$ in $\mathbf{R}^2$).

5. Let $A = \begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}$. Find the eigenvalues of A.

Solution.

We find the characteristic polynomial $\det(A-\lambda I)$ any way we like. The computation below uses the cofactor expansion along the second row:

$$\det(A-\lambda I) = \det\begin{pmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{pmatrix}$$

$$= (1-\lambda)\cdot\big[(5-\lambda)(-2-\lambda)-3\cdot 6\big] = (1-\lambda)(\lambda^2-3\lambda-28)$$

$$= -\lambda^3+4\lambda^2+25\lambda-28 \quad \text{or} \quad (1-\lambda)(\lambda-7)(\lambda+4)$$

The characteristic equation is thus $(1-\lambda)(\lambda-7)(\lambda+4)=0$, so the eigenvalues are $\lambda=-4$, $\lambda=1$, and $\lambda=7$.

6. Using facts about determinants, justify the following fact: if A is an $n\times n$ matrix, then A and A^T have the same characteristic polynomial.

Solution.

We will use three facts which apply to all $n\times n$ matrices B, Y, Z:

(1) $\det(B)=\det(B^T)$.

(2) $(Y-Z)^T = Y^T - Z^T$

(3) If λ is any scalar then $(\lambda I)^T = \lambda I$ since the identity matrix is completely symmetric about its diagonal.

Using these three facts in order, we find

$$\det(A-\lambda I) = \det\big((A-\lambda I)^T\big) = \det\big(A^T-(\lambda I)^T\big) = \det(A^T-\lambda I).$$