Math 1553 Supplement §4.2, 4.3

Solutions

1. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

a) Is it possible that T is one-to-one? If yes, justify why. If no, find distinct vectors v and w so that $T(v) = T(w)$.

b) Is it possible that T is onto? Justify your answer.

Solution.

a) From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

The corresponding vector equation is just

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ both satisfy $Av = Aw = 0$, so T cannot be one-to-one.

b) Yes. If $\{v_1, v_3, v_4\}$ is linearly independent then A will have a pivot in every row and T will be onto. Such a matrix A is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}.$$

2. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (y, y)$.

b) JUST FOR FUN: Consider T : (Smooth functions) $\to$ (Smooth functions) given by $T(f) = f'$ (the derivative of f). Then T is not a transformation from any $\mathbf{R}^n$ to $\mathbf{R}^m$, but it is still *linear* in the sense that for all smooth f and g and all scalars c (by properties of differentiation we learned in Calculus 1):

$$T(f+g) = T(f) + T(g) \quad \text{since} \quad (f+g)' = f' + g'$$

$$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

Is T one-to-one?

Solution.

a) This is not onto. Everything in the range of T has its first coordinate equal to its second, so there is no (x,y,z) such that $T(x,y,z)=(1,0)$. It is not one-to-one: for instance, $T(0,0,0)=(0,0)=T(0,0,1)$.

b) T is not one-to-one. If T were one-to-one, then for any smooth function b, the equation $T(f)=b$ would have at most one solution. However, Note that if f and g are the functions $f(t)=t$ and $g(t)=t-1$, then f and g are different functions but their derivatives are the same, so $T(f)=T(g)$. Therefore, T is not one-to-one. It is not within the scope of Math 1553. If you find it confusing, feel free to ignore it.

3. In each case, determine whether T is linear. Briefly justify.

a) $T(x_1,x_2)=(x_1-x_2,x_1+x_2,1)$.

b) $T(x,y)=(y,x^{1/3})$.

c) $T(x,y,z)=2x-5z$.

Solution.

a) Not linear. $T(0,0)=(0,0,1)\neq(0,0,0)$.

b) Not linear. The $x^{1/3}$ term gives it away. $T(0,2)=(0,2^{1/3})$ but $2T(0,1)=(0,2)$.

c) Linear. In fact, $T(v)=Av$ where

$$A=\begin{pmatrix}2 & 0 & -5\end{pmatrix}.$$

4. For each matrix A, describe what the associated matrix transformation T does to $\mathbf{R}^3$ geometrically.

$$\textbf{a)}\ \begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\qquad \textbf{b)}\ \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$$

Solution.

a) We compute

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\-y\\z\end{pmatrix}.$$

This is the reflection over the xz-plane.

b)

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}y\\x\\0\end{pmatrix}.$$

This is projection onto the xy-plane, followed by reflection over the line $y=x$.

5. Let's go back to the 4.2-4.3 worksheet problem #3. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, and $(1,1,1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy- plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane.

In the worksheet, we found the matrix for the transformation T caused by the wolf. Geometrically describe the image of the house under T.

Solution.

In the worksheet, we found $T(x) = Ax$ where

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We know the house has been effectively destroyed, but what do its remains look like? To get an idea, let's look at what happens to the vertices.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

This indicates the pyramid has been squashed into a triangle in the xy-plane with vertices $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$. (the point $\begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$ is along the top side of this triangle).

Effectively, the pyramid was rotated and then destroyed, so that its (rotated) base is all that remains.