Math 1553 Supplement, §3.1 and §3.2

Solutions

1. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 1 & -3 & -4 & -9 \\ 3 & -1 & 8 & 9 \end{array}\right)$$

Question: Does the corresponding linear system have a solution? If so, what is the solution set?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

f) Find a **different** solution in parts (e) and (d).

Solution.

a) What are the solutions to the following vector equation?

$$x\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + z\begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}$$

b) What is the solution set of the following linear system?

$$\begin{aligned} 2x - 2y + 2z &= 0 \\ x - 3y - 4z &= -9 \\ 3x - y + 8z &= 9 \end{aligned}$$

c) There exists a solution if and only if $\begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}$ is in Span $\left\{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}\right\}$.

e) Row reducing yields

$$\left(\begin{array}{ccc|c} 1 & 0 & 7/2 & 9/2 \\ 0 & 1 & 5/2 & 9/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence z is a free variable, so the solution in parametric form is

$$\begin{aligned} x &= \tfrac{9}{2} - \tfrac{7}{2}z \\ y &= \tfrac{9}{2} - \tfrac{5}{2}z. \end{aligned}$$

Taking $z = 0$ yields the solution $x = y = 9/2$.

f) Taking $z = 1$ yields the solution $x = 1, y = 2$.

2. Let $v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} \qquad w = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$.

Question: Is w a linear combination of v_1 and v_2? In other words, is w in Span$\{v_1, v_2\}$?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) Formulate this question as an augmented matrix.

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) Does the following vector equation have a solution?

$$x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$$

b) Does the following linear system have a solution?

$$\begin{aligned} 2x - 2y &= 2 \\ x - 3y &= -4 \\ 3x - y &= 8 \end{aligned}$$

c) As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & -2 & 2 \\ 1 & -3 & -4 \\ 3 & -1 & 8 \end{array}\right)$$

e) Row reducing yields

$$\left(\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{array}\right)$$

so $x = 7/2$ and $y = 5/2$.

3. Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Is b in the span of the columns of A? In other words, is b a linear combination of the columns of A? Justify your answer.

Solution.

Let v_1, v_2, and v_3 be the columns of A. We are asked to determine whether there are scalars x_1, x_2, and x_3 so that $x_1v_1 + x_2v_2 + x_3v_3 = b$, which means

$$\begin{aligned} x_1 \qquad\quad + 5x_3 &= \;\;2 \\ -2x_1 + \;\; x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= \;\;6 \end{aligned}$$

We translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The right column is not a pivot column, so the system is consistent. Therefore, b is in the span of the columns of A (in other words, b is a linear combination of the columns of A).

We weren't asked to solve the equation explicitly, but if we wanted to do so, we would use the RREF of the matrix above to write

$$x_1 = 2 - 5x_3 \qquad x_2 = 3 - 4x_3 \qquad x_3 = x_3 \;\; (x_3 \text{ is free}).$$

In fact, we can take $x_1 = 2, x_2 = 3$, and $x_3 = 0$, to write

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

4. Consider the vector equation

$$x\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + z\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}.$$

Question: Is there a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -2 & 3 & -5 \\ 1 & -1 & 0 & -1 \\ 3 & -1 & 4 & -2 \end{array}\right)$$

b) What is the solution set of the following linear system?

$$\begin{aligned} 2x - 2y + 3z &= -5 \\ x - y \phantom{{}+ 3z} &= -1 \\ 3x - y + 4z &= -2 \end{aligned}$$

c) There exists a solution if and only if $\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}$ is in Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right\}$.

e) Row reducing yields

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

so $x = 3/2$, $y = 5/2$, and $z = -1$.

5. Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

a) Every set of four or more vectors in $\mathbf{R}^3$ will span $\mathbf{R}^3$.

b) The span of any set contains the zero vector.

Solution.

a) This is **false.** For instance, the vectors

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\end{pmatrix}\right\}$$

only span the x-axis.

b) This is **true.** We have

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.$$

Aside: the span of the empty set is equal to $\{0\}$, because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector v, you get $v+$(no other summands), which is just v; and the only vector which gives you v when you add it to v, is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)