

## Section 3.3

### Matrix Equations

# Matrix $\times$ Vector

the first number is  
the number of rows

the second number is  
the number of columns

Let  $A$  be an  $m \times n$  matrix

$$A = \left( \begin{array}{c|c|c|c} | & | & \cdots & | \\ \hline v_1 & v_2 & \cdots & v_n \\ \hline | & | & \cdots & | \end{array} \right) \quad \text{with columns } v_1, v_2, \dots, v_n$$

## Definition

The **product** of  $A$  with a vector  $x$  in  $\mathbf{R}^n$  is the linear combination

$$Ax = \left( \begin{array}{c|c|c|c} | & | & \cdots & | \\ \hline v_1 & v_2 & \cdots & v_n \\ \hline | & | & \cdots & | \end{array} \right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

Annotations:  
- "this means the equality is a definition" points to the definition symbol.  
- "these must be equal" points to the  $x_n$  in the vector and the  $x_n v_n$  term.

The output is a vector in  $\mathbf{R}^m$ .

Note that the number of **columns** of  $A$  has to equal the number of **rows** of  $x$ .

## Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

# Matrix Equations

An example

## Question

Let  $v_1, v_2, v_3$  be vectors in  $\mathbf{R}^3$ . How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

**Answer:** Let  $A$  be the matrix with columns  $v_1, v_2, v_3$ , and let  $x$  be the vector with entries  $2, 3, -4$ . Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$

# Matrix Equations

In general

Let  $v_1, v_2, \dots, v_n$ , and  $b$  be vectors in  $\mathbf{R}^m$ . Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_n v_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \left( \begin{array}{c|c|c|c} | & | & \cdots & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{array} \right) \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if  $A$  is any  $m \times n$  matrix, then

$$Ax = b \quad \text{is equivalent to the} \quad x_1 v_1 + x_2 v_2 + \dots + x_n v_n = b$$

vector equation

where  $v_1, \dots, v_n$  are the columns of  $A$ , and  $x_1, \dots, x_n$  are the entries of  $x$ .

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$\begin{aligned}2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5\end{aligned}$$

2. As an augmented matrix:

$$\left( \begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array} \right)$$

3. As a vector equation ( $x_1 v_1 + \cdots + x_n v_n = b$ ):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ( $Ax = b$ ):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

In particular, *all four have the same solution set.*

# Matrix $\times$ Vector

Another way

## Definition

A **row vector** is a matrix with one row. The product of a row vector of length  $n$  and a (column) vector of length  $n$  is

$$(a_1 \quad \cdots \quad a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} a_1 x_1 + \cdots + a_n x_n.$$

This is a scalar.

If  $A$  is an  $m \times n$  matrix with rows  $r_1, r_2, \dots, r_m$ , and  $x$  is a vector in  $\mathbf{R}^n$ , then

$$Ax = \begin{pmatrix} -r_1- \\ -r_2- \\ \vdots \\ -r_m- \end{pmatrix} x = \begin{pmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{pmatrix}$$

This is a vector in  $\mathbf{R}^m$  (again).

# Matrix $\times$ Vector

Both ways

## Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (4\ 5\ 6) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ (7\ 8\ 9) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Note this is the same as before:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Now you have *two* ways of computing  $Ax$ .

In the second, you calculate  $Ax$  one entry at a time.

The second way is usually the most convenient, but we'll use both.

## Spans and Solutions to Equations

Let  $A$  be a matrix with columns  $v_1, v_2, \dots, v_n$ :

$$A = \left( \begin{array}{c|c|c|c} | & | & \cdots & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{array} \right)$$

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$  has a solution

$$\iff \text{there exist } x_1, \dots, x_n \text{ such that } A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

"if and only if"

$$\iff \text{there exist } x_1, \dots, x_n \text{ such that } x_1 v_1 + \cdots + x_n v_n = b$$

$$\iff b \text{ is a linear combination of } v_1, \dots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

The last condition is geometric.



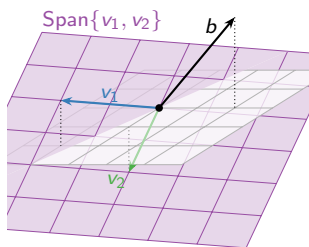
# Spans and Solutions to Equations

## Example

### Question

Let  $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$ . Does the equation  $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$  have a solution?

[interactive]



Columns of  $A$ :

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Target vector:

$$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Is  $b$  contained in the span of the columns of  $A$ ? It sure doesn't look like it.

**Conclusion:**  $Ax = b$  is *inconsistent*.

# Spans and Solutions to Equations

Example, continued

## Question

Let  $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$ . Does the equation  $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$  have a solution?

**Answer:** Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left( \begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array} \right) \xrightarrow{\text{row reduce}} \left( \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

The last equation is  $0 = 1$ , so the system is *inconsistent*.

In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution, as the picture shows.

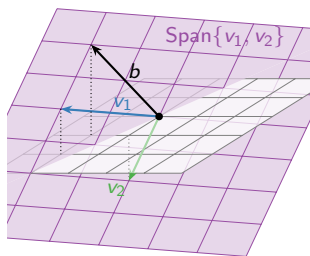
# Spans and Solutions to Equations

## Example

### Question

Let  $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$ . Does the equation  $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  have a solution?

[interactive]



Columns of  $A$ :

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Target vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is  $b$  contained in the span of the columns of  $A$ ? It looks like it: in fact,

$$b = 1v_1 + (-1)v_2 \implies x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

# Spans and Solutions to Equations

Example, continued

## Question

Let  $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$ . Does the equation  $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  have a solution?

**Answer:** Let's do this systematically using row reduction.

$$\left( \begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array} \right) \xrightarrow{\text{row reduce}} \left( \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right)$$

This gives us

$$x = 1 \quad y = -1.$$

This is consistent with the picture on the previous slide:

$$1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{or} \quad A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

## Poll

True or false: (can be done by eyeballing equation)

The matrix equation  $\begin{pmatrix} 1 & -3 & 0 \\ 0 & -1 & 6 \\ 0 & 2 & 3 \end{pmatrix} \mathbf{x} = \mathbf{b}$  is consistent

for every  $\mathbf{b}$  in  $\mathbf{R}^3$ .

- A. True
- B. False

# When Solutions Always Exist

Here are criteria for a linear system to *always* have a solution.

## Theorem

Let  $A$  be an  $m \times n$  (non-augmented) matrix. The following are equivalent:

1.  $Ax = b$  has a solution for all  $b$  in  $\mathbf{R}^m$ .
2. The span of the columns of  $A$  is all of  $\mathbf{R}^m$ .
3.  $A$  has a pivot in each row.

recall that this means  
that for given  $A$ , either they're  
all true, or they're all false

Why is (1) the same as (2)? This was the Very Important box from before.

Why is (1) the same as (3)? If  $A$  has a pivot in each row then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \quad \text{and } (A | b) \quad \begin{pmatrix} 1 & 0 & * & 0 & * & | & * \\ 0 & 1 & * & 0 & * & | & * \\ 0 & 0 & 0 & 1 & * & | & * \end{pmatrix}.$$

reduces to this:

There's no  $b$  that makes it inconsistent, so there's always a solution. If  $A$  doesn't have a pivot in each row, then its reduced form looks like this:

$$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be} \quad \begin{pmatrix} 1 & 0 & * & 0 & * & | & 0 \\ 0 & 1 & * & 0 & * & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 16 \end{pmatrix}.$$

made  
inconsistent:

# When Solutions Always Exist

Continued

## Theorem

Let  $A$  be an  $m \times n$  (non-augmented) matrix. The following are equivalent:

1.  $Ax = b$  has a solution for all  $b$  in  $\mathbf{R}^m$ .
2. The span of the columns of  $A$  is all of  $\mathbf{R}^m$ .
3.  $A$  has a pivot in each row.

In the following demos, the **violet** region is the span of the columns of  $A$ . This is the same as the set of all  $b$  such that  $Ax = b$  has a solution.

[example where the criteria are satisfied]

[example where the criteria are not satisfied]

## Properties of the Matrix–Vector Product

Let  $c$  be a scalar,  $u, v$  be vectors, and  $A$  a matrix.

- ▶  $A(u + v) = Au + Av$
- ▶  $A(cv) = cAv$

For instance,  $A(3u - 7v) = 3Au - 7Av$ .

**Consequence:** If  $u$  and  $v$  are solutions to  $Ax = 0$ , then so is every vector in  $\text{Span}\{u, v\}$ . Why?

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(xu + yv) = xAu + yAv = x0 + y0 = 0.$$

(Here  $0$  means the zero vector.)

**Important**

The set of solutions to  $Ax = 0$  is a span.



- ▶ We have four equivalent ways of writing a system of linear equations:
  1. As a system of equations.
  2. As an augmented matrix.
  3. As a vector equation.
  4. As a matrix equation  $Ax = b$ .
- ▶  $Ax = b$  is consistent if and only if  $b$  is in the span of the columns of  $A$ . The latter condition is geometric: you can draw pictures of it.
- ▶  $Ax = b$  is consistent for all  $b$  in  $\mathbf{R}^m$  if and only if the columns of  $A$  span  $\mathbf{R}^m$ .